Recitation 9

Probability

Expected Value

Intuitively, the expected value is the weighted average of values, kind of a mass center of the probability distribution.

More formally, the *expected value* of a random variable is denoted $\mathbb{E}[X]$ and is defined as

$$\mathbb{E}[X] = \sum_{s \in S} X(s) \Pr(s) = \sum_{r \in X(S)} r \Pr(X = r).$$

We define the *conditional expected values* as follows: Given that event E has occurred, the expectation of random variable X is

$$\mathbb{E}[X \mid E] = \sum_{r \in X(S)} r \Pr(X = r \mid E). \tag{1}$$

Moreover, the *linearity of expectation* can be very useful in calculating expected value: Given that Z, X, Y are three random variables defined on a sample space S and a and b are two real numbers such that $Z = aX + bY$, we know that $\mathbb{E}[Z] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ must be true.

Let's practice this through a task:

Task 1

Tim and Joe are playing a game. They flip a fair coin 3 times. When the coin is heads, Joe pays $1 to Tim; and when the coin is tails, Tim pays $1 to Joe.

a. Let G_i be a random variable representing what Tim gains on the i-th round. For instance, $G_3 = -1$ if the coin is tails.

   What is the expected value of G_i?

b. Let G be a random variable that represents Tim's *total* gain in this game. What is the expected value of G?

c. What is the expected value of G if the coin is biased and the probability of heads is p? in other words, generalize your solution from part b in terms of p.

d. Tim and Joe are still using the biased coin from part c. Let H_1 be the event that the first coin is heads. What is $\mathbb{E}[G|H_1]$?

e. Use your answers to calculate $\mathbb{E}[G]$ and $\mathbb{E}[G \mid H_1]$ when $p = 0.7$.

f. Assume that $p = 0.7$ and let's say we want to change the game to make it "fair." If the flip is tails, then Tim pays a dollar to Joe—how much should Joe pay Tim on Heads so that for any number of flips we know $\mathbb{E}[G] = 0$?

Task 2

Tim and Joe are now playing a similar, but different, game. This time they flip a coin **2 times**. Let X be the random variable that is equal to the number of heads and Y the random variable that is equal to the number of tails. At the end of the game, Joe pays Tim X^2 dollars. Once again, let G be the random variable for Tim's gain.

a. Calculate $\mathbb{E}[X]^2$ when the probability of heads is $p = 0.7$.

b. Calculate $\mathbb{E}[G]$ when $p = 0.7$.

c. Does $\mathbb{E}[G] = \mathbb{E}[X]^2$?

d. Find an example that shows that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ does not hold where X and Y are not independent variables.

Hint: Try X and Y as defined above.

e. Prove that if X and Y are two independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Bayes Rule

The Bayes Rule can be summarized as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$.

Task 3

Assume Brown's CS department has an evaluation system for CS courses based on student evaluations. In any class, the students can fill the evaluation form and give a score of $0, 1$, or 2 to the course. Let X be the random variable of this score. The students of CS0220 either like the course with probability $3/4$ (Event L) or they do not like the course with probability $1/4$ (Event $\neg L$).

Assume that the conditional probability distribution of X given L is

$$\Pr(X = 0 \mid L) = 1/8$$

$$\Pr(X = 1 \mid L) = 1/4$$
$$\Pr(X = 2 \mid L) = 5/8$$

and given that they do not like the course ($\neg L$) it is

$$\Pr(X = 0 \mid \neg L) = 9/10$$

$$\Pr(X = 1 \mid \neg L) = 1/10$$
$$\Pr(X = 2 \mid \neg L) = 0.$$

a. If a student has given score of 0 to CS0220, what is the probability that they do not like the course?

b. Use the definition of conditional expected value (Equation 1) and find $\mathbb{E}[X \mid \neg L]$

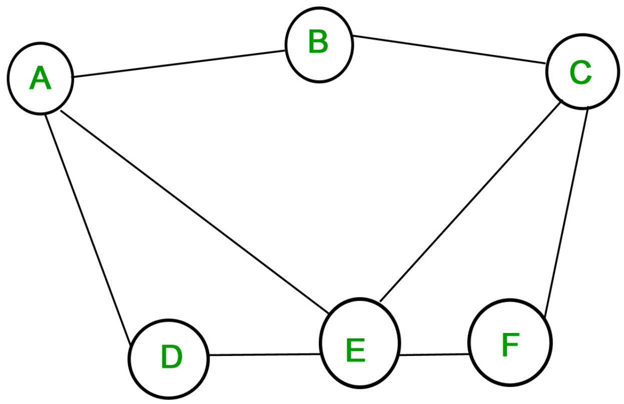
c. *Optional:* Find $\mathbb{E}[X]$.

Checkpoint — Call over a TA!

Introduction to Graph Theory

A **graph** is two related sets: $V(G)$, the vertex set, and $E(G)$, the edge set. Each element of $E(G)$ is a set containing exactly two elements of $V(G)$.

We often visualize graphs by drawing the vertices as dots and the edges as lines between them. Here is an example of a graph with vertex set $\{A, B, C, D, E, F\}$ and edge set $\{\{A, B\}, \{B, C\}, \{A, E\}, \{C, E\}, \{A, D\}, \{C, F\}, \{D, E\}, \{E, F\}\}$.



Note that we have defined an edge as a set of cardinality two: this means that a vertex cannot have an edge to itself, and there is no sense of direction in edges. Additionally, since edges are contained in a set, there can be at most one edge between any two vertices.

- If $u, v \in V(G)$, u is **adjacent to** v if $\{u, v\} \in E(G)$. (Note that by this definition a vertex is not adjacent to itself).
- The **degree** of a vertex is a count of the number of vertices it is adjacent to. Formally, for a vertex v, $\deg(v) = |\{u|\{v, u\} \in E(G)\}|$.
- The **empty graph** on n vertices has an empty edge set.
- The **complete graph** on n vertices K_n has all possible edges between vertices, that is, $E(G) = \{(u, v)|u, v \in E(G), u \neq v\}$. Every pair of vertices is adjacent.
- A **subgraph** G' of a graph G is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Since G' is a graph, each edge in $E(G')$ must be between two vertices in $V(G')$.

Task 4

Let G be a graph such that $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{\{b, c\}, \{a, b\}, \{d, e\}, \{c, d\}, \{d, b\}\}$.

a. Draw G.

b. What is the degree of each vertex in G?

c. Say we take a vertex v uniformly at random from $V(G)$. Let X be a random variable that represents the degree of v. What is the probability distribution of X? That is, for each value in the range of X, what is the probability that X takes that value?

d. What is $\mathbb{E}[X]$?

e. *Optional:* Can you develop a general expression for $\mathbb{E}[X]$ in terms of $|V|$ and $|E|$?

Checkoff — Call over a TA!