Recitation 9
Probability

Expected Value

Intuitively, the expected value is the weighted average of values, kind of a mass center of the probability distribution.

More formally, the expected value of a random variable is denoted $\mathbb{E}[X]$ and is defined as

$$\mathbb{E}[X] = \sum_{s \in S} X(s) \Pr(s) = \sum_{r \in \mathbb{X}(S)} r \Pr(X = r).$$

We define the conditional expected values as follows: Given that event $E$ has occurred, the expectation of random variable $X$ is

$$\mathbb{E}[X \mid E] = \sum_{r \in \mathbb{X}(S)} r \Pr(X = r \mid E). \quad (1)$$

Moreover, the linearity of expectation can be very useful in calculating expected value: Given that $Z$, $X$, $Y$ are three random variables defined on a sample space $S$ and $a$ and $b$ are two real numbers such that $Z = aX + bY$, we know that $\mathbb{E}[Z] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ must be true.

Let’s practice this through a task:

Task 1

Tim and Joe are playing a game. They flip a fair coin 3 times. When the coin is heads, Joe pays $1 to Tim; and when the coin is tails, Tim pays $1 to Joe.

a. Let $G_i$ be a random variable representing what Tim gains on the $i$-th round. For instance, $G_3 = -1$ if the coin is tails.

What is the expected value of $G_i$?

b. Let $G$ be a random variable that represents Tim’s total gain in this game. What is the expected value of $G$?
c. What is the expected value of $G$ if the coin is biased and the probability of heads is $p$? in other words, generalize your solution from part b in terms of $p$.

d. Tim and Joe are still using the biased coin from part c. Let $H_1$ be the event that the first coin is heads. What is $E[G|H_1]$?

e. Use your answers to calculate $E[G]$ and $E[G | H_1]$ when $p = 0.7$.

f. Assume that $p = 0.7$ and let’s say we want to change the game to make it “fair.” If the flip is tails, then Tim pays a dollar to Joe—how much should Joe pay Tim on Heads so that for any number of flips we know $E[G] = 0$?

Task 2

Tim and Joe are now playing a similar, but different, game. This time they flip a coin 2 times. Let $X$ be the random variable that is equal to the number of heads and $Y$ the random variable that is equal to the number of tails. At the end of the game, Joe pays Tim $X^2$ dollars. Once again, let $G$ be the random variable for Tim’s gain.

a. Calculate $E[X]^2$ when the probability of heads is $p = 0.7$.

b. Calculate $E[G]$ when $p = 0.7$. 

c. Does $E[G] = E[X]^2$?


d. Find an example that shows that $E[XY] = E[X]E[Y]$ does not hold where $X$ and $Y$ are not independent variables.

Hint: Try $X$ and $Y$ as defined above.


e. Prove that if $X$ and $Y$ are two independent random variables, then $E[XY] = E[X]E[Y]$. 
Bayes Rule

The Bayes Rule can be summarized as

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

where \( A \) and \( B \) are events and \( P(B) \neq 0 \).

Task 3

Assume Brown’s CS department has an evaluation system for CS courses based on student evaluations. In any class, the students can fill the evaluation form and give a score of 0, 1, or 2 to the course. Let \( X \) be the random variable of this score. The students of CS0220 either like the course with probability \( 3/4 \) (Event \( L \)) or they do not like the course with probability \( 1/4 \) (Event \( \neg L \)).

Assume that the conditional probability distribution of \( X \) given \( L \) is

\[
\begin{align*}
\Pr(X = 0 \mid L) &= 1/8 \\
\Pr(X = 1 \mid L) &= 1/4 \\
\Pr(X = 2 \mid L) &= 5/8
\end{align*}
\]

and given that they do not like the course (\( \neg L \)) it is

\[
\begin{align*}
\Pr(X = 0 \mid \neg L) &= 9/10 \\
\Pr(X = 1 \mid \neg L) &= 1/10 \\
\Pr(X = 2 \mid \neg L) &= 0.
\end{align*}
\]

a. If a student has given score of 0 to CS0220, what is the probability that they do not like the course?

b. Use the definition of conditional expected value (Equation 1) and find \( \mathbb{E}[X \mid \neg L] \)

Checkpoint — Call over a TA!
A graph is two related sets: $V(G)$, the vertex set, and $E(G)$, the edge set. Each element of $E(G)$ is a set containing exactly two elements of $V(G)$.

We often visualize graphs by drawing the vertices as dots and the edges as lines between them. Here is an example of a graph with vertex set \{A, B, C, D, E, F\} and edge set \{{A, B}, {B, C}, {A, E}, {C, E}, {A, D}, {C, F}, {D, E}, {E, F}\}.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
D & \rightarrow & E \\
\downarrow & & \downarrow \\
C & \rightarrow & F \\
\end{array}
\]

Note that we have defined an edge as a set of cardinality two: this means that a vertex cannot have an edge to itself, and there is no sense of direction in edges. Additionally, since edges are contained in a set, there can be at most one edge between any two vertices.

- If $u, v \in V(G)$, $u$ is adjacent to $v$ if $\{u, v\} \in E(G)$. (Note that by this definition a vertex is not adjacent to itself).

- The degree of a vertex is a count of the number of vertices it is adjacent to. Formally, for a vertex $v$, $\deg(v) = |\{u|\{v, u\} \in E(G)\}|$.

- The empty graph on $n$ vertices has an empty edge set.

- The complete graph on $n$ vertices $K_n$ has all possible edges between vertices, that is, $E(G) = \{(u, v)|u, v \in E(G), u \neq v\}$. Every pair of vertices is adjacent.

- A subgraph $G'$ of a graph $G$ is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Since $G'$ is a graph, each edge in $E(G')$ must be between two vertices in $V(G')$. 
Task 4
Let $G$ be a graph such that $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{\{b, c\}, \{a, b\}, \{d, e\}, \{c, d\}, \{d, b\}\}$.

a. Draw $G$.

b. What is the degree of each vertex in $G$?

c. Say we take a vertex $v$ uniformly at random from $V(G)$. Let $X$ be a random variable that represents the degree of $v$. What is the probability distribution of $X$? That is, for each value in the range of $X$, what is the probability that $X$ takes that value?

d. What is $\mathbb{E}[X]$?

e. Optional: Can you develop a general expression for $\mathbb{E}[X]$ in terms of $|V|$ and $|E|$?

Checkoff — Call over a TA!