Recitation 8

Counting and Intro to Probability

Probability

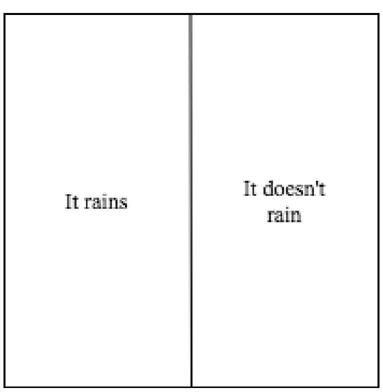
A **finite discrete probability space** is a pair $(S, \Pr)$ for some finite set S and some function $\Pr : S \to \mathbb{R}$, where, for all $x \in S$, $\Pr(x) > 0$, and $\sum_{x \in S} \Pr(x) = 1$. The set S is called the *sample space*, and the elements of S are called *outcomes*. The function Pr is called the *probability distribution function*.

Where do these defintions come from? It's all about predictions. The sample space S is supposed to represent a set of atomic outcomes—that is, the set of all things that can happen. All these outcomes are mutually exclusive.

An *event* is a subset of the sample space S, containing some atomic outcomes. For example, if I flip a coin 3 times, the outcomes are HHH, TTT, ..., so the sample space is $S = \{\text{HHH}, \text{TTT}, \text{TTH}, \dots\}$, with $2^3 = 8$ outcomes. An event can be: *We see exactly 1 H*, which we view as a set of outcomes $\{\text{HTT}, \text{THT}, \text{TTH}\}$. An event can also be the empty set: for example, the event *We see 4 Hs* is the empty set, since it is not a possible outcome (it does not correspond to anything in the sample space). Similarly, the event *We see an elephant* is the empty set for this probability space.

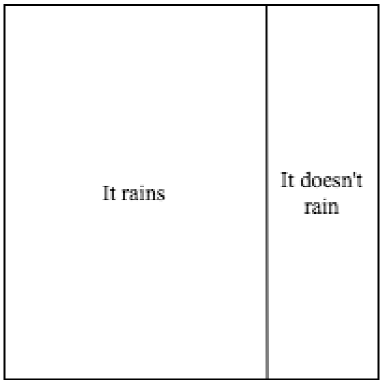
How many different events can the above sample space have? Since each event is a subset of the sample space, the number of events is equal to the number of possible subsets, which here is 2^8 (since the sample space has 8 elements).

To understand the probability distribution function, let's think about our sample space as a box of area 1. Then, each outcome occupies a certain amount of *area* within the sample space. That is, the area is *distributed* among the outcomes according to the probability distribution function! The larger the area, the more likely the outcome is.



In this example, we have that each outcome occupies the same amount of area in the box—each is equally likely. Because we have 2 outcomes, we have that the probability of each outcome is $\frac{1}{2}$.

Let's take a look at another way we could divide up the area of the box among the outcomes.



Here, the probability of it raining is $\frac{3}{4}$, so this outcome takes up $\frac{3}{4}$ of the area within the box. The remaining amount of area is $\frac{1}{4}$, which becomes the probability that it doesn't rain.

Task 1

a. If we have n outcomes in S, and Pr assigns each outcome an equal amount of area within our box, what is the probability of a particular outcome?

   Note: Such a division of the space is called a *uniform distribution*!

b. Consider $S = \{\text{It's sunny}, \text{It's raining}\}$, where the probability distribution is 0.8 and 0.2, respectively. Under what constraints on the world is this sample space reasonable?

c. Is the following probability distribution valid? Experiment: flipping a biased coin. $S = \{H, T\}$, $\Pr(H) = 1/3$, $\Pr(T) = 1/3$.

The Relationship Between Counting and Probability

Questions about probability are often closely tied to questions about counting. Often, we need to count two things—the set of outcomes in our event, and the set of total possible outcomes. In a uniform distribution, the probability of getting an event E is simply $\Pr(E) = |E|/|S|$.

Task 2

a. Suppose S is the set of all binary strings of length n, and Pr is a uniform distribution. What is the probability of getting a string with k ones (where $0 \leq k \leq n$)?

b. For what value(s) of k is the probability of getting a string with k ones the largest? Try it out for some values and see if you find a pattern.

c. Explain how we can use n-ary (not just binary) strings to help us with other problems that involve, say, flipping a fair coin m times, or rolling a die m times.

Checkpoint 1 — call over a TA!

Conditional Probability and Independence

We sometimes want to know what the probability of something is, given that we know an outcome from certain subset of outcomes has happened (that is, no outcome outside that subset can happen).

Warmup

Prof. Lewis has 2 coins: one fair, and one double heads. He picks one uniformly at random (that is, each coin has probability of 1/2 of being picked), but he doesn't tell us which one he picks. Prof. Lewis then flips the coin he picks, and he then shouts out the result.

Suppose Prof. Lewis shouts out "Heads." What is the probability he flipped the fair coin, given that we know the coin flip resulted in heads (H)?

Hint: How many ways can the coin flip result in heads, and how many of them start with the fair coin?

We can generalize our work here. Given two events $A, B \subseteq S$ (the sample space) where $\Pr(B) > 0$, we define the *conditional probability of A given B* as

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

We know that the only possible outcomes are those in B (since B is supposed to have happened). We therefore make $\Pr(B)$ our denominator to *renormalize* our universe.

Looking at this formula, we also get a general definition for $Pr(A \cap B)$:

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A) = \Pr(B)\Pr(A|B)$$

This is a derivation of **Bayes' Theorem**, and it will be your friend in the probability unit.

There are certain special situations where $\Pr(B|A) = \Pr(B)$; that is, where the fact that A happened does not affect the probability of B happening. We have a name for this situation.

Definition: Two events $A, B \subseteq S$ are *(pairwise) independent* if

$$\Pr(A|B) = \Pr(A)$$

If $\Pr(B) = 0$, B and any other event are independent.

Equivalently, A and B are independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Note that these two definitions are equivalent—try to convince yourself by substituting the definition of conditional probability!

Task 3

a. If two events A and B have an empty intersection, what is the probability of A AND B?

b. For each of the following pairs of events A and B, identify whether they are independent, and justify why or why not.

   i. Suppose we roll a fair die, and suppose A = rolling an even number, and B = rolling a number greater than three.

   ii. *Optional:* Suppose we flip a fair coin three times, and suppose A = the last coin is a tails, and B = there is a run of exactly two tails (that is, two, but not three, tails are flipped in a row).

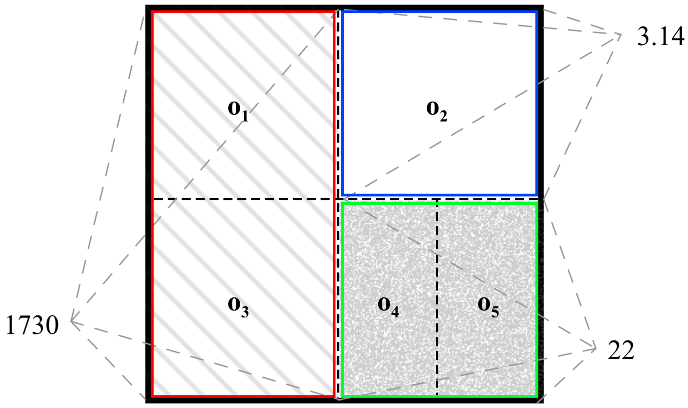
c. *Optional:* If two events A and B are *mutually exclusive*, are they *independent*? You can assume $\Pr(A)$ and $\Pr(B)$ are nonzero.

Random Variables

From a probability space (S, p), we can create a new probability space (S', p'), where S' is a partition of S, that is, $S' = \{P_1, P_2, ...P_n\}$, and where $p' = \Pr(P_i)$ for all i.

Visually, we can take a box partitioned into m outcomes, and then partition these m outcomes into n groups. The amount of area each group takes up is just the sum of the outcomes within the group, so this box with m groups is just like having a probability space with m outcomes.

This leads us to one of the biggest, most famous misnomers in all of mathematics: random variables. A **random variable** is a function from a set of outcomes S to $\mathbb{R}$ or $\mathbb{Z}$. We can think of this random variable as partitioning S, where each group is a set of outcomes that are all assigned the same value. We can also think of the random variable as assigning each group a different value. This diagram shows what we mean:



Here, $o_1, \ldots, o_5$ are outcomes in S. Our random variable assigns two of them to 1730, one of them to 3.14, and two of them to 22. Thus, the random variable partitions the outcomes into 3 groups. Let's walk through some other more concrete examples of random variables.

Task 4

Let random variable X be on the coin flip sample space $\{H, T\}$, where $X(H) = 1$, and $X(T) = 0$. (This random variable is also known as the indicator random variable of event H.)

a. If $\Pr(H) = 1/2$, and $\Pr(T) = 1/2$, then what is $\Pr(X = 1)$? How about $\Pr(X = 0)$?

b. We could also have the random variable Y, where the domain of Y is S, all sequences of coin flips of length n, and where $Y(s) =$ the number of heads in s.

   If S is uniformly distributed, then what is $\Pr(Y = k)$, where $0 \leq k \leq n$?

   Hint: You computed this value in last week's recitation.

c. Let C_i be a random variable for the ith coin flip, s_i, in our sequence, s, of coin flips of length n, where $C_i(H) = 1$ and $C_i(T) = 0$. Let $C(s) = C_1(s_1) + C_2(s_2) + \ldots C_n(s_n)$. Explain why $C(s) = Y(s)$.

d. *Optional:* Explain why $\Pr(C = k) = \binom{n}{k} \Pr(C_i = 1)^k \Pr(C_i = 0)^{n-k}$.

e. *Optional:* Given an expression for $\Pr(C \geq k)$.

f. *Optional:* Compute $\Pr(C = k | C_1 = 1)$.

Checkpoint 2 — Call over a TA!

Task 5

The Brown Review is a test-prep company that publishes books helping high school students prepare for the upcoming Accelerated Placement tests. Recently, they published a study with 20000 students nationwide showing that using their books to prepare for the AP Statistics exam resulted were 5% more likely to pass than those who studied using their rival, Karron's. However, The Brown Review did not publish all of their data, and you uncover the following data table:

	The Brown Review	Karron's	Total
Took a statistics class	$\frac{8550}{9000} = 95\%$	$\frac{4450}{4500} = 99\%$	$\frac{13000}{13500} = 96\%$
Did not take a statistics class	$\frac{750}{1000} = 75\%$	$\frac{4350}{5500} = 79\%$	$\frac{5100}{6500} = 78\%$
Total	$\frac{9300}{10000} = 93\%$	$\frac{8800}{10000} = 88\%$	$\frac{18100}{20000} = 90\%$

a. Make an argument as to why The Brown Review appears to be a better choice for test prep.

b. Make an argument as to why Karron's is the better choice.

c. Given that it is very likely that a student who takes a class before the exam succeeds more often than students who did not take a class, how could the Brown Review manipulate sample sizes such that their final percentage looks higher?

This contradiction of conclusions is known as Simpson's Paradox, which you can read more about here. This occurs when comparisons of two variables in separate groups yield one conclusion, but comparisons of the variables overall yield a different result due to a more significant trend working in the background, like how students taking or not taking the class determines outcomes.

d. It is very common for news services to read the conclusion of a scientific study and report the final results directly. It is also extremely common for readers to look at a news headline and not read the article. How do these practices encourage misinformation? Why do details of a study matter?

e. With the growing popularity of data analysis and machine learning, our society increasingly relies on statistical tools to explain the world we live in. Why is it extremely important to never take any conclusions we garner from these methods at face value? What are the dangers in becoming over-reliant on these methods to solve complex problems such as policing or policymaking?

Task 6

Anastasio goes to PVDonuts and gets four boxes of donuts for everyone. One box has two chocolate donuts, two boxes have one chocolate and one glazed, and one box has two glazed donuts.

a. Anastasio picks a box at random, and gives a random donut to Justin. If that donut is chocolate, what is the probability the other donut in the box is glazed?

b. Let's consider another scenario, where Justin picks a donut box at random. Before he opens it, Anastasio tells him one of the donuts in the box is chocolate.

   What is the probability the other donut is glazed?

   Hint: Your answer should be different from part a.

Optional: **Task 7**

Suppose that, in a certain family, the probability of each child being born with a cat allergy is 1/2. You can assume one of the parents have it and it gets inherited with 1/2 probability. Assume this family has two children, Tyler and Ben.

a. What is the probability that both Tyler and Ben have a cat allergy?

b. Consider a scenario where you go to the family's house and meet one of the children. They tell you they have a cat allergy, but not their name. What is the probability the other child is allergic?

c. The child you meet says their name is Tyler. Does the probability of the other child being allergic change?

d. Given that at least one of the children is allergic and was born on a Tuesday, what is the probability the family has 2 allergic children? You may assume that the probability a child is born on a given day of the week is 1/7.

Checkoff — Call over a TA!