Recitation 4

Relations and Functions

Part 1: Relations

Definitions

Defn 1: A *relation* $R : A \to B$ is defined on a domain A, co-domain B, and a graph that is a subset of the Cartesian product $A \times B$. As a slight abuse of notation, we will often write the relation R as a subset of $A \times B$. A relation *on* a set A is $R : A \to A$.

Defn 2: An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

Defn 4: A relation R on A is *reflexive* if $\forall a \in A$, $(a, a) \in R$.

Defn 5: A relation R on A is *symmetric* if $\forall (a, b) \in R$, $(b, a) \in R$. An equivalent definition is that a relation is *not* symmetric if $\exists (a, b) \in R$ such that $(b, a) \notin R$.

Defn 6: A relation R on A is *antisymmetric* if $\forall a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$.

Defn 7: A relation R on A is *transitive* if $\forall (a, b), (b, c) \in R$, $(a, c) \in R$. An equivalent definition is that a relation is *not* transitive if $\exists (a, b), (b, c) \in R$ such that $(a, c) \notin R$.

Defn 8: Let R be an equivalence relation on A. Then, the *equivalence class* of $a \in A$, denoted $[a]_R$, is $\{x \in A \mid (a, x) \in R\}$. That is, $[a]_R$ is all of the elements to which a is related.

What is the point of an equivalence relation, anyway?

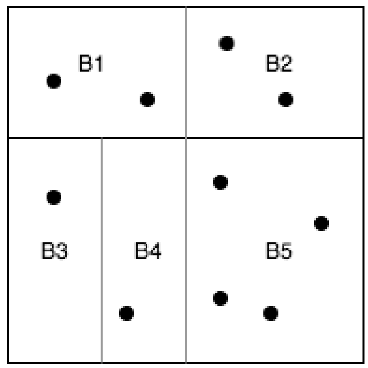
What does it mean for two things to be equal? It can depend on context. For example, you probably generally think of the numbers 2 and 4 as not being equal. However, maybe I want to consider the numbers 2 and 4 to be equal in some contexts because they are both even. We could be in a situation where we only want there to be two kinds of things: even things and odd things. We don't care about anything else like how big or how small the thing is.

An equivalence relation allows us to specify what things in the world are equal to each other, and what things aren't.

An equivalence relation R splits up our world into categories, or equivalence classes. In a given equivalence class, all things within the class are things we consider equal, or equivalent, in the context of R.

For instance, the equivalence relation $R = \{(x, y) \mid x \bmod 2 = y \bmod 2\}$, for all $x, y \in \mathbb{Z}$, divides the world into two categories: the odd and even numbers.

We call the way the equivalence relation splits up our world a *partition*. A little more formally, a *partition* of a set A is a list of subsets $B_1, \ldots, B_k$ of A such that every element of A is in some subset B_i (exhaustive), but no two subsets share an element (mutually exclusive).



One possible partition of some set A, where the dots in the square are distinct elements of A

Task 1

1. Consider the set $A = \{1, 2, 3\}$. In the following questions, all relations are on A. It may be helpful to draw out a diagram of each relation.

   a. $R = A \times A$. List out the elements of R. Is R an equivalence relation? If so, state its equivalence class(es).

   > $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. It is an equivalence relation because everything is related to each other, since it's the entire Cartesian product. There is only one equivalence class: the entire set.

   b. $R = \{(1, 2), (2, 1)\}$. Is this relation *transitive*?

   > No, it's not transitive. $(1, 2), (2, 1) \in R \implies (1, 1) \in R$, but it's not actually in R. The same argument works for $(2, 2)$.

   c. $R = \{(1, 2), (2, 1), (2, 2), (1, 1)\}$. Is this relation reflexive? Symmetric? Transitive?

(3, 3) is not in R, so it's not reflexive (since we defined R to be on A. However, it is transitive and symmetric.

d. If the relation in question iii is not an equivalence relation, can you add one pair to it and make it an equivalence relation? Write the equivalence classes of the new relation.

Yes, add $(3, 3)$. In that case, the equivalence classes become $\{1, 2\}$ and $\{3\}$.

2. Let $A = \{1, 2\}$ and answer to the following questions.

a. What is the equivalence relation on A with the smallest number of equivalence classes possible?

$A \times A$, a world where everything is equal to everything.

b. What is the equivalence relation on A with the largest number of equivalence classes possible?

$\{(1, 1), (2, 2)\}$, where things are equal only if they are truly equal.

c. Is $R_0 = \{\}$ a relation on A?

Yes, the empty set is a subset of every set

d. Is R_0 symmetric? Is it antisymmetric? Why or why not?

Yes, it does not violate our definition of symmetry/antisymmetry as there are no pairs to begin with. ("vacuously true" based on logical equivalence $(F \implies P) \equiv T$)

e. Is R_0 transitive? Why or why not?

Yes, it does not violate our second definition of transitive as there are no pairs. ("vacuously true based on the logical equivalence $(F \implies P) \equiv T$")

f. R_0 is not an equivalence relation. Why?

It is not reflexive.

3. Suppose R is an equivalence relation on S, and $R = \{\}$. What is S?

The empty set, otherwise it is not reflexive.

4. Consider the set B of all students at Brown. For each of the following relations on B, state whether they are reflexive, symmetric, antisymmetric, transitive, or some combination of them. If it is an equivalence relation, then determine the equivalence classes of the relation.

   a. Two students are related if they have the same astrology sign.

   Reflexive, symmetric, and transitive. Therefore equivalence relation. Equivalence classes are students of same astrology sign.

   However, not antisymmetric.

   b. s_1 and s_2 are students and $(s_1, s_2) \in R$ if s_1 is younger than or the exact same age as s_2. (You can assume no students were born at the exact same time.)

   Transitive and anti-symmetric and reflexive, but not symmetric.

   c. Two students are related if they are studying anthropology.

   Symmetric and transitive but not reflexive or anti-symmetric.

   d. Two students are related if they go to Brown.

   Reflexive, symmetric, and transitive, but not antisymmetric. Therefore equivalence relation. One equivalence class which consists of all students at Brown.

Checkpoint 1 — Call a TA over!

Part 2: Functions

Definitions

As a reminder:

Defn 1: A relation $R : X \to Y$ is a **function** if for every x in the domain X, x is mapped to one and only one y in Y, the codomain. Note that in the book this is called a *total function*, and function refers to a *partial function*, where for every x in the domain X, x is mapped to zero or one y in the codomain Y. In this class, we will use function to mean total function and partial function to mean partial function.

Defn 2: The **range** of a function f consists of all members of the codomain of f that are mapped to by some member of the domain of f. It is the *image* of the domain.

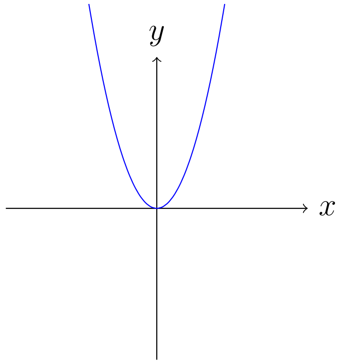
Defn 3: $f : X \to Y$ is **injective (one-to-one)** if, for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently, for any $x, y \in Y$ we have $f(x) = f(y) \implies x = y$, and you can also use its contrapositive $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Defn 4: $f : X \to Y$ is **surjective (onto)** if, for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. For surjective functions, the range is equal to co-domain.

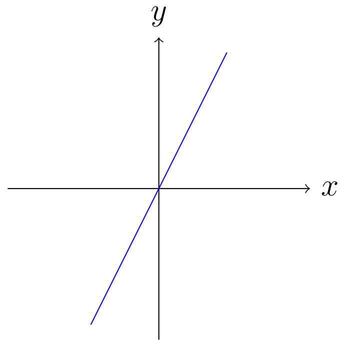
Defn 5: $f : X \to Y$ is a **bijection** if it is both an injection and surjection.

Task 2

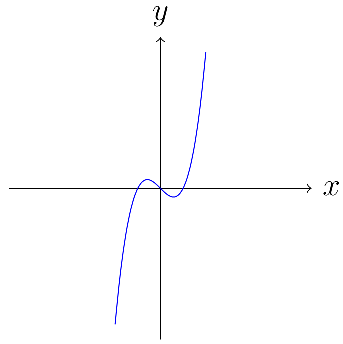
Consider the following functions and determine if the given function is an injection, surjection, and/or bijection.



(a) $f(x) = x^2$



(b) $g(x) = x/2$



(c) $h(x) = x^3 - x$

Discuss your answers!

a. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Not injective or surjective.

b. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{x}{2}$

Injective and surjective. Therefore bijective.

c. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^3 - x$

Surjective, but not injective

d. Question: All of the above functions are defined on $\mathbb{R}$. Consider their graphs in the coordinate system. Which of the following implies surjectivity, which implies injectivity, and which implies neither?

(1) any vertical line intersects the graph **at most** once

(2) any horizontal line intersects the graph **at most** once

(3) any vertical line intersects the graph **at least** once

(4) any horizontal line intersects the graph **at least** once

Surjection: (4). Injection: (2). Neither: (1), (3)

e. Recall the functions defined in parts a-c. Is $f \circ h : \mathbb{R} \to \mathbb{R}$ surjective and/or injective? (Use a graphing calculator if you need to.)

$f \circ h(x) = (x^3 - x)^2$, not surjective, not injective

f. Is $h \circ f : \mathbb{R} \to \mathbb{R}$ surjective and/or injective?

$h \circ f(x) = x^6 - x^2$, not surjective, not injective

g. Let $f : \mathbb{R} \to \mathbb{Z}$ give as output the greatest integer less than or equal to x, denoted as the floor function $f(x) = \lfloor x \rfloor$. For instance, $f(3.5) = 3$, $f(3) = 3$, and $f(\pi) = 3$.[1]

Surjective, but not injective.

h. $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = \lfloor x \rfloor$

Injective and surjective. Therefore bijective.

i. *Optional:* f : Brown University Students → Countries in the World

f(student) = country where student is from

Not injective, and sadly not surjective.

j. *Optional:* f : First Year Students → First Year Dorms

f(student) = dorm that student lives in

Not injective. Presumably surjective.

[1]Note that we can similarly define the ceiling function $f : \mathbb{R} \to \mathbb{Z}$ that gives as output the smallest integer greater than or equal to x, denoted the ceiling function $f(x) = \lceil x \rceil$. For example, $hf(3.5) = 4$, $f(3) = 3$, and $f(\pi) = 4$.

Task 3

Let $A = \{0, 1, 2\}$ and the function $f : \mathcal{P}(A) \to \{0, 1\}^3$.

Denote the output as (f_0, f_1, f_2), where each f_i is 0 or 1. Given an element $X \in \mathcal{P}(A)$, define $f(X)$ where each f_i is 1 iff $i \in X$, and 0 otherwise.

For instance, $f(\{0, 2\}) = (1, 0, 1)$, $f(A) = (1, 1, 1)$, and $f(\emptyset) = (0, 0, 0)$.

a. Consider a set of size n where $A = \{1, 2, \ldots, n\}$. Generalize the above bijective function for $\mathcal{P}(A)$ and binary strings of length n (that is, $\{0, 1\}^n$).

> $f : \mathcal{P}(A) \to \{0, 1\}^n$, let $X \in \mathcal{P}(A)$, for any $i = 1, 2, \ldots, n$, the i-th element in $f(X)$ is 1 iff $i \in X$.

b. Prove that the above function is a bijection.

> **Surjectivity.** Given any $(f_0, f_1, \ldots, f_n)$, we can construct a subset X of A such that $f(X)$ results in the above binary string. For each element $i \in A$, include it in X iff $f_i = 1$. Then, by the definition of our function, we will result in the exact tuple that we started out with, proving surjectivity.
>
> **Injectivity.** We'll show that $X \neq Y \implies f(X) \neq f(Y)$. If the two subsets are not equal, it means that there is at least some element e that one has but the other does not. Without loss of generality, say $e \in X$ but $e \notin Y$.
>
> As a result, $f_e = 1$ for $f(X)$ but $f_e = 0$ for $f(Y)$, so the resulting tuples are not equal. Therefore, we've shown injectivity.
>
> With both properties shown, f is a bijective function.

Optional Task 4

a. If a function $f : X \to Y$ is injective, what can we say about the cardinalities of X and Y? Try making some diagrams where X has more elements than Y, fewer elements than Y, or the same number of elements as Y. When are you able to create an injection, and when are you not?

> If and only if $|X| \leq |Y|$, then there exists $f : X \to Y$ such that f is injective.

b. If a function $f : X \to Y$ is surjective, what can we say about the cardinalities of X and Y? Again, you might want to draw out some examples.

If and only if $|X| \geq |Y|$, then there exists $f : X \to Y$ such that f is surjective.

c. Based on what you've found in the previous two questions, if a function $f : X \to Y$ is bijective, what can we say about the cardinalities of X and Y? When can we create a bijection between two sets, and when can we not?

If and only if $|X| = |Y|$, then there exists $f : X \to Y$ such that f is bijective.

Checkpoint 2 — Call a TA over!

Part 3: Induction

Induction

Most of next week's recitation will focus on induction. But we've introduced the concept this week, and wanted to get an early start on practicing it!

Why does induction work?

Let's consider an infinite ladder (the best kind of ladder). Suppose we can prove to you both of the following things:

1. You can get to the 1^{st} step of the ladder by stepping up to it.
2. If you can get to the k^{th} step of the ladder, then you can get to step $k + 1$ by stepping up to it.

Why is it the case that for all $n \geq 1$, you can get to the n^{th} step of the ladder? Discuss with your group.

> We already know we can get to the first step from the first statement. Then, we know we can get to the second step from the second statement. From there, the process repeats and we conclude that we can get to the third, then the fourth... and so on.

Why are we talking about climbing infinite ladders? Well, it turns out this is a good way to think about how induction works.

The *base case* says that we can reach the first step of the ladder.

The *inductive hypothesis* says that we can get to the k^{th} step of the ladder.

The *inductive step* says that if we can get to the k^{th} step of the ladder, then we can get to step $k + 1$.

Therefore, once we get to step 1, we can get to step 2. Once we get to step 2, we can get to step 3. And so on for all steps of the infinite ladder.

Induction Template

We will now review the template for an inductive proof.

For example, say we are trying to prove that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all $n \in \mathbb{N}$. In other words, show that this is the equation for calculating the sum of squares $0^2 + 1^2 + 2^2 + \cdots + n^2$.

Predicate. Define the predicate $P(n)$. Recall that a predicate is a function that takes in an argument, n, and evaluates to true or false.

Let $P(n)$ be the predicate that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Introduce Induction. Make the aspirational assertion that, for all $n \geq a$, where a is the smallest value we are considering, $P(n)$ holds. Remember to bound n!

We will show that, for all $n \geq 0$, $P(n)$ holds.

Base Case. Show that the base case is true. For some proofs, we may want multiple base cases, but not this time.

We will first show $P(0)$ is true, that

$$\sum_{i=0}^{0} i^2 = 0 \quad \textit{and} \quad \frac{0(0+1)(2*0+1)}{6} = 0$$

so they are equal.

Inductive Hypothesis. State the inductive hypothesis. In standard induction[2], we assume

$P(k)$ is true for some fixed, arbitrary integer $k \geq a$, where a is your base case value. Sometimes, you may need multiple base cases, and you'll want k to be greater than or equal to the biggest of them.

Assume $P(k)$ is true for some fixed, arbitrary integer $k \geq 0$.

Inductive Step. Show that $P(k+1)$ is true given the inductive hypothesis. At some point, you'll want to "invoke the inductive hypothesis", which is using the fact that $P(k)$ is true to show something else in your proof.

[2]We will also cover strong induction, in which we assume $P(i)$ is true for **all** $a \leq i \leq k$

We will now show that $P(k+1)$ holds, namely

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

We know that

$$\sum_{i=0}^{k+1} i^2 = \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2$$

Invoking the inductive hypothesis, we know that

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Therefore

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

as needed.

Conclusion. Conclude your induction.

Because the base case $P(0)$ holds, and because $P(k) \to P(k+1)$, we have shown by the principle of induction that for all $n \geq 0$, $P(n)$ holds. □

⋆ Note ⋆

For the sake of time, we're only going to look for proof *sketches* in recitation. It's alright to not write down everything, as long as you understand it. In your homework, we'll be looking for full-fledged formal proofs.

Task 5

Prove by induction that, for all $n \geq 2$,

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

Let P(n) be the predicate that

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})\cdots(1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

defined for integers $n \geq 2$.

Base Case:
$LHS = 1 - \frac{1}{2^2} = \frac{3}{4}$.
$RHS = \frac{2+1}{4} = \frac{3}{4}$.
Hence, P(2) holds.

Inductive Hypothesis: Suppose that for some fixed, arbitrary integer $k \geq 2$, $P(k)$ holds. That is, $(1 - \frac{1}{2^2})\cdots(1 - \frac{1}{k^2}) = \frac{k+1}{2k}$.

Inductive Step: Now, consider $(1 - \frac{1}{2^2})\cdots(1 - \frac{1}{k^2})(1 - \frac{1}{(k+1)^2})$.

By the I.H., we know that it is equal to $\frac{k+1}{2k}(1 - \frac{1}{(k+1)^2})$.

Distributing, this is equal to $\frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2}$, which simplifies into $\frac{(k+1)^2-1}{2k(k+1)}$.

Expanding the numerator gives us $\frac{k^2+2k}{2k(k+1)}$, and cancelling out k results in $\frac{k+2}{2(k+1)}$, as needed. Thus, P($k+1$) is true.

Conclusion: Since P(2) is true and for any integer $k \geq 2$, and $P(k)$ implies $P(k+1)$, $P(n)$ is true for all integers $n \geq 2$.

Checkoff - Call a TA over!