Recitation 3

Set Equivalence, Relations and Functions

Set Theory: Review

Recall the definitions and terminology for set theory:

Defn 1: A **set** is a collection of objects with no repetition or order.

Defn 2: B is a **subset** of A if every element in B is also in A. This is written as $B \subseteq A$.

Defn 3: $\mathcal{P}(A)$, called the **power set** of A, is the set of all subsets of A.

Defn 4: $A \cup B$ denotes the **union** of sets A and B. This contains all the elements from A, and all of the elements from B.

Defn 5: $A \cap B$ denotes the **intersection** of sets A and B. This contains only the elements that appear in both of the sets.

Defn 6: $A \setminus B$ denotes the **difference** of sets A and B. This contains elements that appear in A but not B. In other words, $A \setminus B = \{x \in A \mid x \notin B\}$. We can read this symbol as "minus."

Defn 7: $\overline{A}$ denotes the **complement** of A relative to some universal set U. $\bar{A} = U \setminus A$, that is, it is everything except what is in A.

Defn 8: $|A|$ denotes the **cardinality** of A, which is a count of the number of elements contained in A.

Set Equivalence

Set Element Method

Defn 1: Two sets S and T are equal if and only if they have the same elements:

$$S = T \iff (\forall x : x \in S \iff x \in T)$$

Defn 2: S and T are equal if and only if both S is a subset of T and T is a subset of S.

From these two definitions, we can construct our set-element method for proving set equalities; that is, we show that two sets are equal if and only if every element of S is also an element of T and every element of T is also an element of S.

In other words, to prove that one set is a subset of the other, we can show that $S \subseteq T$ for arbitrary sets S and T using the following steps:

1. Let x be an element of S.
2. Prove that x is an element of T.
3. Conclude that $S \subseteq T$.

Example

Claim: $A \cap (A \cup B) = A$.

Proof: We show that both $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$.

1. We will first prove our sub-claim that $A \cap (A \cup B) \subseteq A$.

   Consider any $x \in A \cap (A \cup B)$. This means that $x \in A$ and $x \in A \cup B$. In particular, we know that $x \in A$. Thus, we have shown that $A \cap (A \cup B) \subseteq A$.

2. We will then prove the second sub-claim that $A \subseteq A \cap (A \cup B)$.

   Consider any $x \in A$. Then, $x \in A \cup B$. Thus, we know that $x \in A$ and $x \in A \cup B$, which can be rewritten as $x \in A \cap (A \cup B$. This shows that $A \subseteq A \cap (A \cup B)$.

Therefore, $A = A \cap (A \cup B)$.

Now, let's practice!

Task 2

For each of these statements, either prove (using the 'set element' method) or disprove the claim.

Hint: Use a counterexample to disprove a claim!

a. For any two sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

> *Claim*: For all sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
>
> *Proof*:
>
> We will use the set element method to prove our claim. We will first show $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, and we will then show $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.
>
> Consider some x in $\mathcal{P}(A \cap B)$. If x is in $\mathcal{P}(A \cap B)$, then x must be a subset of $A \cap B$. By the definition of intersection, x is a subset of A and x is a subset of B. This means x must be a member of $P(A)$ and x must be a member of $P(B)$. Therefore, x must be a member of $P(A) \cap P(B)$. Thus, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
>
> Now, consider some y in $\mathcal{P}(A) \cap P(B)$. By the definition of intersection, y is in both $\mathcal{P}(A)$ and $\mathcal{P}(B)$. If y is in both $\mathcal{P}(A)$ and $\mathcal{P}(B)$, then y is a subset of both A and B. Thus, $\mathcal{P}(A \cap B)$ must also contain y. Thus, $\mathcal{P}(A) \cap P(B) \subseteq \mathcal{P}(A \cap B)$.
>
> Because $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ and $\mathcal{P}(A) \cap P(B) \subseteq \mathcal{P}(A \cap B)$, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap P(B)$.

b. For any two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

> To disprove our claim, it suffices to provide a counterexample. Let A be $\{1\}$. Let B be $\{3\}$.
>
> $\mathcal{P}(A \cup B) = \{\{1\}, \{3\}, \{1, 3\}, \varnothing\}$.
>
> $\mathcal{P}(A) = \{\{1\}, \varnothing\}$.
>
> $\mathcal{P}(B) = \{\{3\}, \varnothing\}$.
>
> Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{3\}, \varnothing\}$, which is not $\{\{1\}, \{3\}, \{1, 3\}, \varnothing\}$. Thus $\mathcal{P}(A \cup B)$ is not equal to $\mathcal{P}(A) \cup \mathcal{P}(B)$.

Task 3

a. Let A and B be subsets of some universal set. Then $A \setminus (A \setminus B) = A \cap B$.

Let A and B be subsets of some universal set. We will prove that $A \setminus (A \setminus B) = A \cap B$ by proving that $A \setminus (A \setminus B) \subseteq A \cap B$ and that $A \cap B \subseteq A \setminus (A \setminus B)$.

1. We will show that $A \setminus (A \setminus B) \subseteq A \cap B$.

   Let $x \in A \setminus (A \setminus B)$. This means that $x \in A$ and $x \notin (A \setminus B)$.

   We know that an element is in $(A \setminus B)$ if an only if it is in A and not in B. Since $x \notin (A \setminus B)$, we conclude that $x \notin A$ or $x \in B$. However, we also know that $x \in A$ and so we conclude that $x \in B$. This proves that $x \in A$ and $x \in B$.

   This means that $x \in A \cap B$, and hence we have proved that $A \setminus (A \setminus B) \subseteq A \cap B$.

2. We will show that $A \cap B \subseteq A \setminus (A \setminus B)$.

   Now, we choose $y \in A \cap B$. This means that $y \in A$ and $y \in B$.

   We note that $y \in (A \setminus B)$ if and only if $y \in A$ and $y \notin B$ and hence, $y \notin (A \setminus B)$ if and only if $y \notin A$ or $y \in B$. Since we know that $y \in B$, we conclude that $y \notin (A \setminus B)$. Since $y \in A$ and $y \notin (A \setminus B)$, we can conclude that $y \in A \setminus (A \setminus B)$.

   This proves that if $y \in A \cap B$, then $y \in A \setminus (A \setminus B)$ and hence, $A \cap B \subseteq A \setminus (A \setminus B)$.

Since we have proved that $A \setminus (A \setminus B) \subseteq A \cap B$ and $A \cap B \subseteq A \setminus (A \setminus B)$, we conclude that $A \setminus (A \setminus B) = A \cap B$.

b. *Optional:* Let A and B be the following sets:

$$A = \{6n : n \in \mathbb{Z}\}$$
$$B = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$$

Prove that $A = B$.

We will show that both $A \subseteq B$ and $B \subseteq A$.

1. First, we prove that $A \subseteq B$; that is, $\{6n\text{: n} \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

   Choose any $m \in \{6n\text{: } n \in \mathbb{Z}\}$. By definition, we can write $m = 6k$ for some $k \in \mathbb{Z}$.

   In particular, $m = 2(3k)$, so that $m \in \{3n\text{: } n \in \mathbb{Z}\}$ as well. By the definition of intersection, we can conclude that $m \in \{2n\text{: } n \in \mathbb{Z}\} \cap \{3n\text{: } n \in \mathbb{Z}\}$ as required.

2. Next, we prove that $\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n\text{: n} \in \mathbb{Z}\}$.

   Choose any $m \in \{2n\text{: } n \in \mathbb{Z}\} \cap \{3n\text{: } n \in \mathbb{Z}\}$, so that both $m \in \{2n\text{: } n \in \mathbb{Z}\}$ and $m \in \{3n\text{: } n \in \mathbb{Z}\}$.

   In particular, we can find integers j and k such that $m = 2j$ and $m = 3k$. The fact that $m = 2j$ means that m is even, and so $m = 3k$ is even. But the product of two integers is odd if and only if both intgers are odd; since 4 is odd but $3k$ is even, we conclude that k is even. Then, $k = 2a$ for some integer a, and so $m = 3k = 3(2a) = 6a$, which shows that $m \in \{6n\text{:}n \in \mathbb{Z}\}$ as required.

Therefore, we can conclude that $A = B$ as we have shown both $A \subseteq B$ and $B \subseteq A$.

c. *Optional:* Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

1. We will show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

   Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. So $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$. In the former case, $(x, y) \in A \times B$, and in the latter case $(x, y) \in A \times C$. So $(x, y) \in (A \times B) \cup (A \times C)$. Since (x, y) was chosen arbitrarily, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
2. We will show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

   Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$. So $x \in A$ and $y \in B \cup C$, and hence $(x, y) \in A \times (B \cup C)$. Since (x, y) was chosen arbitrarily, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Therefore, we can conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Checkpoint 2 — Call over a TA!

Set Algebra

We can also prove that two sets are equivalent using set algebra. More concretely, we can use the stated laws of set algebra to convert one side of the equation to the other (or convert both sides to an identical expression).

These are sometimes called "rewrite rules," since we're looking for the pattern on one side, and rewriting it to the pattern on the other side. A list of set rewrite rules is available on our course website: https://brown-cs22.github.io/resources/math-resources/sets.pdf

Example (from the sample proofs on our website):

Prove that $(A \cap B) \cup (A \setminus B) = A \cap (B \cup (A \setminus B))$.

$(A \cap B) \cup (A \setminus B)$
$= (A \cap B) \cup (A \cap \overline{B})$ (Set Difference Law)
$= (A \cap (B \cup \overline{B})$ (Distribution)
$= A \cap U$ (Complement Law)
$= A$ (Identity Law)
$= A \cap (A \cup B)$ (Absorption)
$= A \cap (B \cup A)$ (Commutivity)
$= A \cap ((B \cup A) \cap U)$ (Identity Law)
$= A \cap ((B \cup A) \cap (B \cup \overline{B})$ (Complement Law)
$= A \cap (B \cup (A \cap \overline{B}))$ (Distribution)
$= A \cap (B \cup (A \setminus B))$ (Set Difference Law)

Therefore, the equality holds since all these steps are biconditionally true.

Task 4

a. Let A and B be subsets of some universal set U. Prove that $(A \cap \overline{B}) \cup B = A \cup B$ using set algebra.

$(A \cap \overline{B}) \cup B$
$= (A \cup B) \cap (\overline{B} \cup B)$ (Distributive Law)
$= (A \cup B) \cap U$ (Complement Law)

$= A \cup B$ (Identity Law)

b. Let A and B be subsets of some universal set U. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ using set algebra.

$(A \setminus B) \cup (B \setminus A)$

$= (A \cap \overline{B}) \cup (B \cap \overline{A})$ (Set Difference Law)

$= (A \cap \overline{B} \cup B) \cap ((A \cap \overline{B} \cup \overline{A})$ (Distributive Law)

$= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A}))$ (Distributive Law)

$= ((A \cup B) \cap U) \cap (U \cap \overline{B} \cup \overline{A})$ (Complement Law)

$= (A \cup B) \cap (\overline{B} \cup \overline{A})$ (Identity Law)

$= (A \cup B) \cap (\overline{A \cap B})$ (DeMorgan's Law)

$= (A \cup B) \setminus (A \cap B)$ (Set Difference Law)

c. *Optional:* Show that $(A \setminus B) \setminus (B \setminus C) = A \setminus B$.

$(A \setminus B) \setminus (B \setminus C)$

$= (A \cap \overline{B}) \cap (\overline{B \cap \overline{B}})$ (Set Difference Law)

$= (A \cap \overline{B}) \cap (\overline{B} \cup C)$ (DeMorgan's Law)

$= ((A \cap \overline{B}) \cap \overline{B}) \cup ((A \cap \overline{B}) \cap C)$ (Distributive Law)

$= (A \cap (\overline{B} \cap \overline{B})) \cup (A \cap (\overline{B} \cap C))$ (Associative Law)

$= (A \cap \overline{B}) \cup (A \cap (\overline{B} \cap C))$ (Idempotent Law)

$= A \cap (\overline{B}) \cup (\overline{B} \cap C))$ (Distributive Law)

$= A \cap \overline{B}$ (Absorption Law)

$= A \setminus B$ (Set Difference Law)

$= (A \cap \overline{B} \cup B) \cap ((A \cap \overline{B} \cup \overline{A})$ (Distributive Law)

$= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A}))$ (Distributive Law)

$= ((A \cup B) \cap U) \cap (U \cap \overline{B} \cup \overline{A})$ (Complement Law)

$= (A \cup B) \cap (\overline{B} \cup \overline{A})$ (Identity Law)

$= (A \cup B) \cap (\overline{A \cap B})$ (DeMorgan's Law)

$$= (A \cup B) \setminus (A \cap B) \qquad \text{(Set Difference Law)}$$

Checkpoint 2 — Call over a TA!

Google Ad Personalization

Task 5

a. Read this article on Google's ad personalization.

b. If you have a Google account that you actively use, follow the instructions on this page and look at the groups that Google believes you are a part of. If you have ad personalization enabled and feel comfortable doing so, skim your entire ad profile. Did anything surprise you? Why or why not?

> *Pass condition:* Students should be able to identify (if they opted to look at their profile, which is optional) certain groups or trends Google believes they are a part of. Any sort of reaction is fine, as it is impossible to predict how many groups google has put them.

c. **G**oogle defines your ad profile as the intersection of several traits, or sets. This profile will determine what ads you are shown - advertisers choose what demographic groups and interest groups they want their ads to target (you can learn more about interest targeting here and demographic targeting here). Think about the assumptions and biases that come with putting people into sets. What problems and inaccuracies could arise by defining what content individuals receive solely based on the sets they belong to?

> *Pass condition:* Students should be able to identify at least one disadvantage with defining people's content consumption by wide sets, such as stereotyping, inaccuracy, or confirmation bias. They should be able to understand that not every member of a set is the same, but designing programs around what a set "believes" runs the risk of treating every element in that set as interchangeable.
>
> *Examples:*
>
> - I think being able to define which demographic ads are marketed towards could be really dangerous in terms of stereotyping. By defining your product to be a "female" or "male" product it can perpetuate stereotypes about gender roles, like advertising barbies to females and Nerf guns to males. This is not to mention the dangerous "other" category which binds all non-conforming identities together.
> - I think it's unfair to assume that 18-34 year old's all have the same interests, it's kinda weird that its a trait people can market towards.

Grouping people by sets doesn't really leave any nuance between people, as the advertiser will treat every 18-34 year old as if they were the same person.

- Using groups to determine ads seems weird to me. Google's profile just have some weird inaccuracies, and basing content solely on these inaccurate profiles seems like Google would need a lot of data to collect to get accurate results. This scares me since I can't control how much data google gets from my searches and web history.

Prodding questions:

- Let's say an advertiser wants to market to middle-class people for their product. Would it be fair to say all middle-class people have similar tastes and interests?
- What problems could arise by branding a product, like a toy, to be solely marketable to "males" and "females"?
- When an advertiser states they want to market to a certain group, they necessarily assume people in that group have the same taste. How might this gloss over differences between members in that group?

d. Being at the intersection of several traits often means something drastically different than the sum of those traits. For example, take two traits that tend to be underprivileged: low education level and low income level. How might an upper-class high school dropout's opportunities and livelihood differ from a high school dropout experiencing poverty?

What are some other cases of people who belong to the same set in one instance but experience vastly different amounts of privilege and discrimination based on what other sets they are a member of?

Pass condition: Students should be able to recognize and identify the differences in the situations between the two people in the given example (upper-class high school dropout vs. high school dropout experiencing poverty). They should be able to draw at least two comparisons between people who share a common set(s) but potentially have different life experiences, and students should be able to explain why they chose these comparisons. They should understand the core idea of intersectionality - being a member of two disadvantaged groups has unique struggles. For example, discussing discrimination against black women is not as easy as branding things as either "racism" or "sexism"; their experience is unique.

Examples: Identifying potential differences between two high-school dropouts, when one is upper-class and the other is in poverty:

- Upper-class high school dropout could potentially have financial support from their family, while a dropout experiencing poverty may not have the same luxury
- The two may have had different opportunities before they dropped out, leading to them possessing different skill sets.

Comparisons between people who may share a set but have different experiences in life:

- White man vs. white woman in STEM (discrimination towards women in STEM, belief that women are not as capable)
- Black vs white woman
- Any situation in which the student lists a combination of two social or personal identities and points out discrepancies between being being part of only one or two of the listed groups

Prodding questions:

- Why might an upper-class dropout be more keen on taking financial risks such as investing in the stock market or entrepreneurship?
- Would an upper-class
- Think about historically disadvantaged groups. Why is being a member of two of these groups uniquely worse than being member of only one?

Prodding questions:

- We have mostly discussed sets in terms of larger demographic groups, such as race, gender, age, etc. However, that doesn't mean sets are limited to just these broad categories (think of your Google ad profile). Are there really a finite number of sets that people can belong to?
- Can human experiences truly be counted?
- Is it possible fr people to be defined by just one characteristic? What are the pitfalls of treating others as if they are one-dimensional?

Part 2: Functions

Definitions

Defn 1: A relation $R : X \to Y$ is a **function** if for every x in the domain X, x is mapped to one and only one y in Y, the codomain. Note that in the book this is called a *total function*, and function refers to a *partial function*, where for every x in the domain X, x is mapped to zero or one y in the codomain Y. In this class, we will use function to mean total function and partial function to mean partial function.

Defn 2: The **range** of a function f consists of all members of the codomain of f that are mapped to by some member of the domain of f. It is the *image* of the domain.

Defn 3: $f : X \to Y$ is **injective (one-to-one)** if, for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently, for any $x, y \in Y$ we have $f(x) = f(y) \implies x = y$, and you can also use its contrapositive $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Defn 4: $f : X \to Y$ is **surjective (onto)** if, for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. For surjective functions, the range is equal to co-domain.

Defn 5: $f : X \to Y$ is a **bijection** if it is both an injection and surjection.

Task 2

Let A be the set $\{1, 2, 3\}$. Consider the following relation on A, $R_1 = \{(1, 2), (2, 1)\}$.

1. Is R_1 a function?

> No; not all members of A are mapped to something in A. It is a partial function but not a (total) function.

Now, consider R_2, another relation on A: $\{(1, 2), (2, 1), (3, 2)\}$.

1. Is R_2 a function?

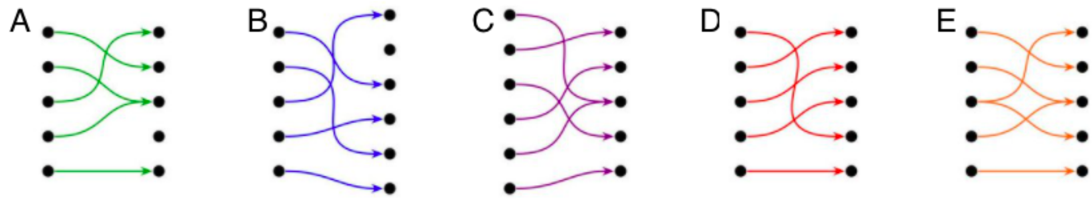
> Yes.

2. If R_2 is a function, what is its codomain? How about its range?

The codomain is A. The range is $\{1, 2\}$.

Task 3

Consider these diagrams that visualize a relation $R : A \to B$. The diagrams have two sets of dots, one for A and one for B, and they have an arrow from a to b in whenever $(a, b) \in R$.



Match each of the five diagrams, labeled A–E, with one of these five descriptions below:

1. ___ Not a function
2. ___ A function that is neither surjective nor injective
3. ___ A surjective function that is not injective
4. ___ An injective function that is not surjective
5. ___ A bijective function — both surjective and injective

1. E is not a function
2. A is a function that is neither surjective nor injective
3. C is a surjective function that is not injective
4. B is an injective function that is not surjective
5. D is a bijective function

Checkoff — Call over a TA!