Recitation 3

Set Equivalence, Relations and Functions

Set Theory: Review

Recall the definitions and terminology for set theory:

Defn 1: A **set** is a collection of objects with no repetition or order.

Defn 2: B is a **subset** of A if every element in B is also in A. This is written as $B \subseteq A$.

Defn 3: $\mathcal{P}(A)$, called the **power set** of A, is the set of all subsets of A.

Defn 4: $A \cup B$ denotes the **union** of sets A and B. This contains all the elements from A, and all of the elements from B.

Defn 5: $A \cap B$ denotes the **intersection** of sets A and B. This contains only the elements that appear in both of the sets.

Defn 6: $A \setminus B$ denotes the **difference** of sets A and B. This contains elements that appear in A but not B. In other words, $A \setminus B = \{x \in A \mid x \notin B\}$. We can read this symbol as "minus."

Defn 7: $\overline{A}$ denotes the **complement** of A relative to some universal set U. $\bar{A} = U \setminus A$, that is, it is everything except what is in A.

Defn 8: $|A|$ denotes the **cardinality** of A, which is a count of the number of elements contained in A.

Set Equivalence

Set Element Method

Defn 1: Two sets S and T are equal if and only if they have the same elements:

$$S = T \iff (\forall x : x \in S \iff x \in T)$$

Defn 2: S and T are equal if and only if both S is a subset of T and T is a subset of S.

From these two definitions, we can construct our set-element method for proving set equalities; that is, we show that two sets are equal if and only if every element of S is also an element of T and every element of T is also an element of S.

In other words, to prove that one set is a subset of the other, we can show that $S \subseteq T$ for arbitrary sets S and T using the following steps:

1. Let x be an element of S.
2. Prove that x is an element of T.
3. Conclude that $S \subseteq T$.

Example

Claim: $A \cap (A \cup B) = A$.

Proof: We show that both $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$.

1. We will first prove our sub-claim that $A \cap (A \cup B) \subseteq A$.

   Consider any $x \in A \cap (A \cup B)$. This means that $x \in A$ and $x \in A \cup B$. In particular, we know that $x \in A$. Thus, we have shown that $A \cap (A \cup B) \subseteq A$.

2. We will then prove the second sub-claim that $A \subseteq A \cap (A \cup B)$.

   Consider any $x \in A$. Then, $x \in A \cup B$. Thus, we know that $x \in A$ and $x \in A \cup B$, which can be rewritten as $x \in A \cap (A \cup B$. This shows that $A \subseteq A \cap (A \cup B)$.

Therefore, $A = A \cap (A \cup B)$.

Now, let's practice!

Task 2

For each of these statements, either prove (using the 'set element' method) or disprove the claim.

Hint: Use a counterexample to disprove a claim!

a. For any two sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

b. For any two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

Task 3

a. Let A and B be subsets of some universal set. Then $A \setminus (A \setminus B) = A \cap B$.

b. *Optional:* Let A and B be the following sets:

$$A = \{6n : n \in \mathbb{Z}\}$$
$$B = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$$

Prove that $A = B$.

c. *Optional:* Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Checkpoint 2 — Call over a TA!

Set Algebra

We can also prove that two sets are equivalent using set algebra. More concretely, we can use the stated laws of set algebra to convert one side of the equation to the other (or convert both sides to an identical expression).

These are sometimes called "rewrite rules," since we're looking for the pattern on one side, and rewriting it to the pattern on the other side. A list of set rewrite rules is available on our course website: https://brown-cs22.github.io/resources/math-resources/sets.pdf

Example (from the sample proofs on our website):

Prove that $(A \cap B) \cup (A \setminus B) = A \cap (B \cup (A \setminus B))$.

$$
\begin{aligned}
&(A \cap B) \cup (A \setminus B) \\
&= (A \cap B) \cup (A \cap \overline{B}) && \text{(Set Difference Law)} \\
&= (A \cap (B \cup \overline{B}) && \text{(Distribution)} \\
&= A \cap U && \text{(Complement Law)} \\
&= A && \text{(Identity Law)} \\
&= A \cap (A \cup B) && \text{(Absorption)} \\
&= A \cap (B \cup A) && \text{(Commutivity)} \\
&= A \cap ((B \cup A) \cap U) && \text{(Identity Law)} \\
&= A \cap ((B \cup A) \cap (B \cup \overline{B}) && \text{(Complement Law)} \\
&= A \cap (B \cup (A \cap \overline{B})) && \text{(Distribution)} \\
&= A \cap (B \cup (A \setminus B)) && \text{(Set Difference Law)}
\end{aligned}
$$

Therefore, the equality holds since all these steps are biconditionally true.

Task 4

a. Let A and B be subsets of some universal set U. Prove that $(A \cap \overline{B}) \cup B = A \cup B$ using set algebra.

b. Let A and B be subsets of some universal set U. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ using set algebra.

c. *Optional:* Show that $(A \setminus B) \setminus (B \setminus C) = A \setminus B$.

Checkpoint 2 — Call over a TA!

Google Ad Personalization

Task 5

a. Read this article on Google's ad personalization.

b. If you have a Google account that you actively use, follow the instructions on this page and look at the groups that Google believes you are a part of. If you have ad personalization enabled and feel comfortable doing so, skim your entire ad profile. Did anything surprise you? Why or why not?

c. **G**oogle defines your ad profile as the intersection of several traits, or sets. This profile will determine what ads you are shown - advertisers choose what demographic groups and interest groups they want their ads to target (you can learn more about interest targeting here and demographic targeting here). Think about the assumptions and biases that come with putting people into sets. What problems and inaccuracies could arise by defining what content individuals receive solely based on the sets they belong to?

d. Being at the intersection of several traits often means something drastically different than the sum of those traits. For example, take two traits that tend to be underprivileged: low education level and low income level. How might an upper-class high school dropout's opportunities and livelihood differ from a high school dropout experiencing poverty?

   What are some other cases of people who belong to the same set in one instance but experience vastly different amounts of privilege and discrimination based on what other sets they are a member of?

   Prodding questions:

   - We have mostly discussed sets in terms of larger demographic groups, such as race, gender, age, etc. However, that doesn't mean sets are limited to just these broad categories (think of your Google ad profile). Are there really a finite number of sets that people can belong to?
   - Can human experiences truly be counted?
   - Is it possible fr people to be defined by just one characteristic? What are the pitfalls of treating others as if they are one-dimensional?

Part 2: Functions

Definitions

Defn 1: A relation $R : X \to Y$ is a **function** if for every x in the domain X, x is mapped to one and only one y in Y, the codomain. Note that in the book this is called a *total function*, and function refers to a *partial function*, where for every x in the domain X, x is mapped to zero or one y in the codomain Y. In this class, we will use function to mean total function and partial function to mean partial function.

Defn 2: The **range** of a function f consists of all members of the codomain of f that are mapped to by some member of the domain of f. It is the *image* of the domain.

Defn 3: $f : X \to Y$ is **injective (one-to-one)** if, for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently, for any $x, y \in Y$ we have $f(x) = f(y) \implies x = y$, and you can also use its contrapositive $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Defn 4: $f : X \to Y$ is **surjective (onto)** if, for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. For surjective functions, the range is equal to co-domain.

Defn 5: $f : X \to Y$ is a **bijection** if it is both an injection and surjection.

Task 2

Let A be the set $\{1, 2, 3\}$. Consider the following relation on A, $R_1 = \{(1, 2), (2, 1)\}$.

1. Is R_1 a function?

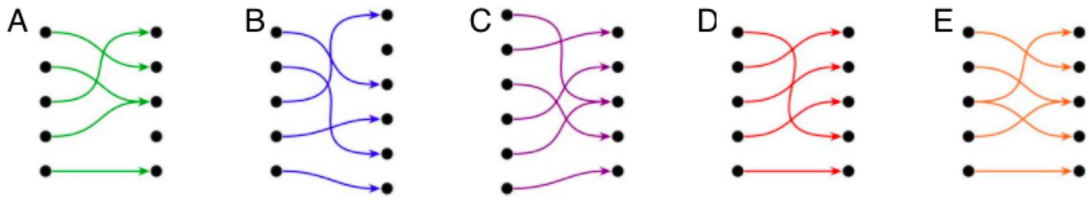
Now, consider R_2, another relation on A: $\{(1, 2), (2, 1), (3, 2)\}$.

1. Is R_2 a function?

2. If R_2 is a function, what is its codomain? How about its range?

Task 3

Consider these diagrams that visualize a relation $R : A \to B$. The diagrams have two sets of dots, one for A and one for B, and they have an arrow from a to b in whenever $(a, b) \in R$.



Match each of the five diagrams, labeled A–E, with one of these five descriptions below:

1. ___ Not a function
2. ___ A function that is neither surjective nor injective
3. ___ A surjective function that is not injective
4. ___ An injective function that is not surjective
5. ___ A bijective function — both surjective and injective

Checkoff — Call over a TA!