Linearity of Expectation

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Overview

Expected winnings

D dollars won, w event that you win, $\overline{w}$ event that you lose.

$\mathbb{E}[D] = \mathbb{E}[D|w]\Pr[w] + \mathbb{E}[D|\overline{w}](1 - \Pr[w])$

Game is *fair* if $\mathbb{E}[D] = 0$.

Example: $\mathbb{E}[D|w] = -\mathbb{E}[D|\overline{w}]$, $\Pr[w] = 1/2$.

Example: SAT points (before March 2016). $\mathbb{E}[D|w] = 1$, $\mathbb{E}[D|\overline{w}] = -1/4$, $\Pr[w] = 1/5$.

$1 \cdot .2 + (-1/4) \cdot .8 = 0$.

Bar bet

Three people. Each player puts in \$2. Each player predicts the outcome of a coin flip. Correct guessers split the pot. If all wrong, money returned.

$$
\begin{aligned}
\mathbb{E}[D] \quad = \quad & \mathbb{E}[D|\text{win alone}]\Pr[\text{win alone}] \\
& + \mathbb{E}[D|\text{win with one}]\Pr[\text{win with one}] \\
& + \mathbb{E}[D|\text{win with both}]\Pr[\text{win with both}] \\
& + \mathbb{E}[D|\text{lose}]\Pr[\text{lose}] \\
& + \mathbb{E}[D|\text{all lose}]\Pr[\text{all lose}]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[D] \quad = \quad & 4 \cdot \Pr[\text{win alone}] \\
& + 1 \cdot \Pr[\text{win with one}] \\
& + 0 \cdot \Pr[\text{win with both}] \\
& - 2 \cdot \Pr[\text{lose}] \\
& + 0 \cdot \Pr[\text{all lose}]
\end{aligned}
$$

Random guesses

$$\begin{aligned}\mathbb{E}[D] \quad = \quad & 4 \cdot \Pr[\text{win alone}] \\ & +1 \cdot \Pr[\text{win with one}] \\ & +0 \cdot \Pr[\text{win with both}] \\ & -2 \cdot \Pr[\text{lose}] \\ & +0 \cdot \Pr[\text{all lose}]\end{aligned}$$

Everyone guesses randomly.

- $\Pr[\text{win alone}] = 1/8$
- $\Pr[\text{win with one}] = 1/4$
- $\Pr[\text{win with both}] = 1/8$
- $\Pr[\text{lose}] = 3/8$
- $\Pr[\text{all lose}] = 1/8$

$4 \cdot 1/8 + 1 \cdot 1/4 + 0 \cdot 1/8 - 2 \cdot 3/8 + 0 \cdot 1/8 = 0$. Fair!

Collusion

$$\begin{aligned}\mathbb{E}[D] \quad = \quad & 4 \cdot \Pr[\text{win alone}] \\ & +1 \cdot \Pr[\text{win with one}] \\ & +0 \cdot \Pr[\text{win with both}] \\ & -2 \cdot \Pr[\text{lose}] \\ & +0 \cdot \Pr[\text{all lose}]\end{aligned}$$

The other two collude to make opposite guesses.

- $\Pr[\text{win alone}] = 0$
- $\Pr[\text{win with one}] = 1/2$
- $\Pr[\text{win with both}] = 0$
- $\Pr[\text{lose}] = 1/2$
- $\Pr[\text{all lose}] = 0$

$4 \cdot 0 + 1 \cdot 1/2 + 0 \cdot 0 - 2 \cdot 1/2 + 0 \cdot 0 = -0.5$. I lose!

Irrational bookies

Three horses, *a*, *b*, *c*. For any horse you can bet that it will win or lose. I give you odds:

- 1:1 odds for *a*
- 3:1 odds against *b*
- 4:1 odds against *c*

That is: you can choose to bet, say, \$60 on *b* winning. If *b* wins, I pay you \$180. If *b* loses, you pay me \$60. Or you can bet \$60 on *b* losing. If *b* wins, you pay me \$60. If *b* loses, I pay you \$20.

Should you bet with me?

Yes—I'm irrational! You should bet: \$100 on *a*, \$50 on *b*, \$40 on *c*. You spend \$190. No matter which horse wins, you win \$200.

Be careful with logical combinations of events, too.

Linearity of expectation theorem

Theorem: For any random variables $R_1, \ldots, R_k$ and constants $a_1, \ldots, a_k \in \mathbb{R}$,

$$\mathbb{E}\left[\sum_{i=1}^{k} a_i R_i\right] = \sum_{i=1}^{k} a_i \mathbb{E}[R_i].$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{k} a_i R_i\right] &= \sum_{\omega \in \mathcal{S}} \Pr(\omega)\left[\sum_{i=1}^{k} a_i R_i(\omega)\right] && \text{defn expt} \\
&= \sum_{i=1}^{k} a_i \sum_{\omega \in \mathcal{S}} \Pr(\omega)[R_i(\omega)] && \text{move sum} \\
&= \sum_{i=1}^{k} a_i \mathbb{E}[R_i]. && \text{defn expt}
\end{aligned}$$

C’mon seven...

Let R_1 and R_2 be random variables corresponding to dice. What’s the expected value of their sum? Outcome space is 36 different possible rolls. Linearity of expectation lets us short that out...

$\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2] = 3.5 + 3.5 = 7$, woo!

Note that this result does *not* depend on independence. It holds even if we make the second die dependent on the first die, as long as it doesn’t interfere with the expectation of the second die. $R_2 = R_1, R_2 = 7 - R_1, R_2 = \text{rem}((R_1 - 1) \times 101, 6) + 1$.

Shuffle 1 to 5

Start with the numbers 1 through 5, in order. Then, choose a random permutation. Look to see how many numbers stayed in their original place.

Example:

1	2	3	4	5	
4	1	5	3	2	0
4	5	3	1	2	1
1	3	4	2	5	2

What's the expected number of numbers that stayed in their original place?

The hard way

Let $F(i)$ be the number of ways a permutation of 5 numbers can be arranged so exactly i numbers are in their original place.

The expected number of numbers that stayed in their original place is $\sum_{i=0}^{5} i\ F(i)/5!$.

- What's $F(5)$? 1.
- What's $F(4)$? 0.
- What's $F(0)$? $4\times$ huh. Turns out this value is the number of "derangements" of 5 items. Gets very tricky.
- What's $F(1)$? It's 5 times the number of derangements of 4 items...

I think we could get there, but it's a hike.

The easy way

Let I_i be an indicator random variable on whether the *i*th number is in the *i*th place.

The answer we are looking for is

$$\mathbb{E}[I_1 + I_2 + I_3 + I_4 + I_5] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3] + \mathbb{E}[I_4] + \mathbb{E}[I_5].$$

$\mathbb{E}[I_i]$ is the probability that a random permutation on 5 numbers puts number i in position i. It's $1/5$. So, the answer to the problem is $1/5 + 1/5 + 1/5 + 1/5 + 1/5 = 1$.

Shuffle 1 through n

Take the numbers 1 to n and scramble them.

Again, expected number of digits in the scrambled version that are in their original place is

$$\mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

Interesting, right? Doesn't depend on n. As n increases, more positions, so less likely a random number will be in its original position. But, as n increases, more *chances* for a number to be in its right position. These forces balance.

All CS22 students hand in a quiz. We shuffle them and redistribute so everyone will grade one quiz. What's the expected number of students who get their own quiz back? One.

Expected number of events

Generalization of this idea.

Theorem: Given a set of events, the expected number of these events that occur is the sum of their probabilities.

That's true even if the events are not independent or if their probabilities sum to more than one, or whatever.

Coin flip example

What's the expected number of heads if we flip a coin with probability p of coming up heads n times?

We know a lot about coin flips and number of ways to get k heads. We *can* work through the algebra and use the definition of expectation to get an answer.

Linearity of expectation is so much more direct! The expected number of heads from each flip is p. So, the expected number of heads in n flips is np. Done!

Coupon collecting

Sometimes a company will run a coupon-collecting promotion. For example, each time you make a purchase, you get a coupon at random. If you collect all n coupons, you win a large prize.

How many purchases should you expect to make to win (collect all n coupons) if coupon types are given uniformly at random at each purchase?

Solving it

Consider a sequence of coupons that has all n types at the end. We want to know the *length* of this sequence.

Let X_i be the sequence of coupons from right *after* the $(i-1)$th unique coupon was received until the ith unique coupon is received.

Examples:

$$\underbrace{3}_{X_1}\quad\underbrace{4}_{X_2}\quad\underbrace{5}_{X_3}\quad\underbrace{3\quad 3\quad 2}_{X_4}\quad\underbrace{2\quad 5\quad 1}_{X_5}$$

The length of the entire coupon sequence is the sum of the lengths of the X_is: $1+1+1+3+3=9$.

The *expected* length of the entire sequence is the sum of the expected lengths of the X_is.

What's Xk?

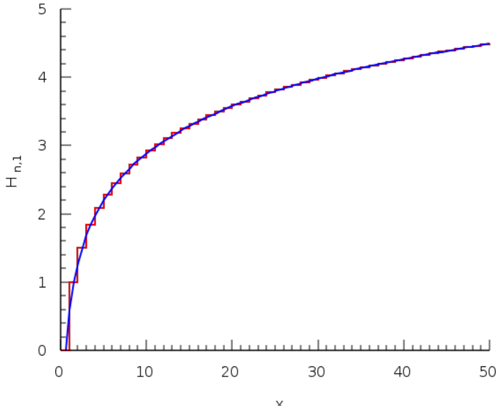
What's the length of X_k? At the beginning of segment k, we have collected $k-1$ different coupons from the n possibilities. Thus, the probability of getting one of the new ones is $p = \frac{n-k+1}{n}$.

What's the expected number of tries until a new one is found? $\frac{n}{n-k+1}$ by the "mean time to failure" analysis!

$\mathbb{E}[\sum_{k=1}^{n} \text{length}(X_k)]$	coup col
$= \sum_{k=1}^{n} \mathbb{E}[\text{length}(X_k)]$	lin of exp
$= \sum_{k=1}^{n} n/(n-k+1)$	mean fail time
$= n \sum_{i=1}^{n} 1/i$	$i = n-k+1$
$= nH(n)$	defn Harmonic number
$\approx n \ln(n)$	prop of H

Aside: harmonic numbers

$$H(n) = \sum_{i=1}^{n} \frac{1}{i} \approx \ln(n)$$



Examples

- People to poll before having representatives of each birthday: $365 \times H(365) \approx 2365$
- Ice skates before you have one of each: $2 \times H(2) = 3$
- Die rolls until you have one of each: $6 \times H(6) = 14.7$
- Number of randomly assigned pigeons until each hole is filled: $n \times H(n) \approx n \ln(n)$