

# Conditional Expectation

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# Overview

- 1 Alternate Definition of Expectation (18.4.4)
- 2 Conditional Expectation (18.4.5)
- 3 Mean Time to Failure (18.4.6)



## Shortcut formula for expectation

Instead of summing over all outcomes, sometimes easier to group outcomes by the blocks defined by the random variable.

$$\mathbb{E}[R] = \sum_{x \in \text{range}(R)} x \cdot \Pr[R = x].$$

### Proof:

$$\begin{aligned} \mathbb{E}[R] &= \sum_{\omega \in \mathcal{S}} R(\omega) \Pr[\omega] && \text{defn expt} \\ &= \sum_{x \in \text{range}(R)} \sum_{\omega \in [R=x]} R(\omega) \Pr[\omega] && \text{range is partition} \\ &= \sum_{x \in \text{range}(R)} \sum_{\omega \in [R=x]} x \Pr[\omega] && \text{defn } R, \omega \\ &= \sum_{x \in \text{range}(R)} x \sum_{\omega \in [R=x]} \Pr[\omega] && \text{factor out } x \\ &= \sum_{x \in \text{range}(R)} x \Pr[R = x]. && \text{defn event prob} \end{aligned}$$

# Median

Not the same as mean or expected value. It's the probability-weighted middle of the set of values.

**Definition:** The *median* of a random variable  $R$  is a value  $x \in \text{range}(R)$  such that

$$\Pr[R \leq x] \geq \frac{1}{2}$$

and

$$\Pr[R \geq x] \geq \frac{1}{2}.$$

## Definition of conditional expectation

**Definition:** The conditional expectation of a random variable  $R$  given event  $A$  is:

$$\mathbb{E}[R|A] := \sum_{r \in \text{range}(R)} r \cdot \Pr[R = r|A].$$

Example:  $R$  is the deck of cards example from before: value of card, 1 for ace, 10 for face.

$$\begin{aligned} & \mathbb{E}[R|R < 10] \\ &= \sum_{i=1}^9 i \Pr[R = i|R < 10] \\ &= \sum_{i=1}^9 i \frac{\Pr[R=i \wedge R < 10]}{\Pr[R < 10]} \\ &= \sum_{i=1}^9 i \frac{1/13}{9/13} \\ &= 1/9 \sum_{i=1}^9 i \\ &= 45/9 = 5. \end{aligned}$$

## The height of elephants

How tall are elephants on average? Don't know. Web search didn't help. Conditional expectations can!

$$\begin{aligned}
 \mathbb{E}[H] &= \mathbb{E}[H|A_f] \cdot \Pr[A_f] + \mathbb{E}[H|A_s] \cdot \Pr[A_s] && \text{split} \\
 &= 10 \cdot 0.9 + \mathbb{E}[H|A_s] \cdot 0.1 && \text{Google} \\
 &= 10 \cdot 0.9 + (\mathbb{E}[H|A_s \wedge m] \Pr[m|A_s] \\
 &\quad + \mathbb{E}[H|A_s \wedge f] \cdot \Pr[f|A_s]) \cdot 0.1 && \text{split} \\
 &= 10 \cdot 0.9 + (9 \cdot 1/2 + 7 \cdot 1/2) \cdot 0.1 && \text{Google} \\
 &= 9.8 && \text{calculator}
 \end{aligned}$$

## Memoryless crash time

A program crashes at the end of each hour of use with probability  $p$ . Let  $C$  be the random variable for the time until a crash occurs. What is  $\mathbb{E}[C]$ ?

Let  $A$  be the event that the program crashes after the first hour and  $\bar{A}$  be the complementary event. Mean time to failure is

$$\mathbb{E}[C] = \mathbb{E}[C|A] \cdot \Pr[A] + \mathbb{E}[C|\bar{A}] \cdot \Pr[\bar{A}]$$

$\mathbb{E}[C|A] = 1$  because  $A$  is the event that the program crashes after one hour.

$\mathbb{E}[C|\bar{A}] = 1 + \mathbb{E}[C]$  because the program runs for an hour, then we're back in the original situation.

$$\mathbb{E}[C] = 1 \cdot p + (1 + \mathbb{E}[C])(1 - p)$$

$$\mathbb{E}[C] = p + 1 + \mathbb{E}[C] - p - p\mathbb{E}[C]$$

$$p\mathbb{E}[C] = 1$$

$$\mathbb{E}[C] = 1/p$$

## Mean time to failure

General principle:

If a system independently fails at each time step with probability  $p$ , then the expected number of steps up to the first failure is  $1/p$ .



## More failures

I'm driving down a street with lots of intersections with traffic lights. Each light is (independently) red 50% of the time and green 50% of the time. How many intersections do I expect to drive through before I stop?

failure hour	red light intersection
$p$	$1/2$
$1/p$	$2$

- Coin flips until heads? 2
- Cards until face card?  $13/3 = 4.3$  ish
- Kobe (0.85) free throws until basket? 1.18