Intro to Probability

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CS 0220 2023

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Overview

1. Probability Spaces (16.5.1)
2. Probability Rules from Set Theory (16.5.2)
3. Uniform Probability Spaces (16.5.3)
4. Infinite Probability Spaces (16.5.4)
Definitions of probability spaces

Definition: A countable sample space $S$ is a nonempty countable(*) set.

Definition: An element $\omega \in S$ is called an outcome.

Definition: A subset of $S$ is called an event.

Definition: A probability function on a sample space $S$ is a function $\Pr : S \rightarrow \mathbb{R}$ such that

1. $\Pr[\omega] \geq 0$ for all $\omega \in S$, and
2. $\sum_{\omega \in S} \Pr[\omega] = 1$.

Definition: A sample space together with a probability function is called a probability space. For any event $E \subseteq S$, the probability of $E$ is defined to be the sum of the probabilities of outcomes in $E$:

$$\Pr[E] := \sum_{\omega \in E} \Pr[\omega].$$
Countability

Side note: a set $S$ is *countable* if it is finite or there is a bijective function $\mathbb{N} \rightarrow S$.

Intuition: we can “list” the elements of $S$. $\{s_0, s_1, s_2, \ldots\}$. Maybe the list ends, maybe it doesn’t...

We’ll mostly deal with finite probability spaces.
Sum rule

Rule: If \( \{E_0, E_1, \ldots, \} \) is collection of disjoint events, then

\[
\Pr \left[ \bigcup_{n \in \mathbb{N}} E_n \right] = \sum_{n \in \mathbb{N}} \Pr[E_n].
\]

Like the sum rule in counting.

Example: I counted plants at the CS22 nursery. 60% of the plants had red flowers, 30% had yellow flowers, and 10% had no flowers. If I pick a plant at random, the probability that it has flowers is 90%.

What is sample space? What are events?
Complement rule

A new day, a new collection of plants:

<table>
<thead>
<tr>
<th>set</th>
<th>type</th>
<th>Pr</th>
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<tbody>
<tr>
<td>$A$</td>
<td>Red flowers</td>
<td>0.60</td>
</tr>
<tr>
<td>$B$</td>
<td>Yellow flowers</td>
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</tr>
<tr>
<td>$A \cap B$</td>
<td>both</td>
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Since we know that the sets $A$ and $\overline{A}$ are disjoint and cover all possibilities, the sum rule tells us that $\Pr[A] + \Pr[\overline{A}] = 1$.

**Rule:** $\Pr[\overline{A}] = 1 - \Pr[A]$.

Example: The chance that a plant does not have red flowers is (in symbols) $\Pr[\overline{A}] = 1 - \Pr[A] = 0.40$. 
Difference Rule

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Rule: \(Pr[B \setminus A] = Pr[B] - Pr[A \cap B]\)

Example: The chance that a plant has red flowers but not yellow flowers is (in symbols)
\(Pr[A \setminus B] = Pr[A] - Pr[A \cap B] = 0.45.\)

Proof: Follows from the Sum Rule because \(B\) is the union of the disjoint sets \(B - A\) and \(A \cap B\).
Inclusion-Exclusion

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**Rule:** $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$

Example: The chance that a plant has flowers is (in symbols) $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] = 0.70$.

Proof: Follows from the Sum and Difference Rules, because $A \cup B$ is the union of the disjoint sets $A$ and $B \setminus A$. 
Boole’s Inequality

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**Rule:** $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$

Example: The chance that a plant has flowers is (in symbols) $\Pr[A \cup B] \leq \Pr[A] + \Pr[B] = 0.85$.

Proof by inclusion-exclusion and the fact that probabilities are non-negative.
Monotonicity Rule

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**Rule:** If $A \subseteq B$, then $\Pr[A] \leq \Pr[B]$

Example: The chance that a plant has yellow flowers must be at least as big as the chance that it has both red and yellow flowers.

Proof: $\Pr[B] = \Pr[A \cup (B \setminus A)] = \Pr[A] + \Pr[B - A] \geq \Pr[A]$. First equality because $A \subseteq B$. Then, they add together by the sum rule because the two sets are disjoint. Then, the last inequality holds because probabilities are non-negative.
Puzzle

You go into a “big box” store near the end of the pandemic. About 50% of the people in your community are vaccinated. You can see that about 40% of the people in the store are wearing masks.

What’s the probability of encountering someone who is neither masked nor vaccinated?
Union bound

Rule:

$$\Pr[E_1 \cup E_2 \cup \cdots \cup E_n] \leq \Pr[E_1] + \Pr[E_2] + \cdots + \Pr[E_n].$$

Example: The probability that a student has conflict with an exam is 0.001. What’s the probability that any of 140 students have a conflict? Can’t assume independence because groups of students take classes together, do sports together. Can’t get an exact answer with the information provided.

Union bounds says probability of student 1 or 2 or 3 ... 140 less than or equal to $140 \times 0.001 = 0.14$.

Used in machine learning all the time.
Uniform

**Definition:** A finite probability space $S$ is said to be *uniform* if $\Pr[\omega]$ is the same for every outcome $\omega \in S$.

In finite spaces, for any $E \subseteq S$,

$$\Pr[E] = \frac{|E|}{|S|}.$$ 

Examples: Sides of a die, cards in a deck.

Contrast with: Vowels vs. consonants, primes vs. composites.
Counting example

What’s the probability that 5 coin flips leads to a palindromic sequence?

What’s the space of possibilities \( S \)? The results of 5 coin flips: HTTHH. \(|S| = 2^5 = 32\).

What’s the event of interest \( E \)? Palindromic results: TTHTT. \(|E| = 2^3 = 8\). That’s because the first 3 flips are “free”, then the 4th flip must match the 2nd and the fifth flip must match the first.

The probability, therefore, is \(|E|/|S| = 2^3/2^5 = 1/2^2 = 1/4\).
Three-sided coin

If we want a uniform distribution over two options, we can flip a coin (H/T). If we want a uniform distribution over four options, we can flip two coins (HH/HT/TH/TT). What if we want a uniform distribution over three options?

We could flip two coins and say it’s option 1 if both heads, option 2 if both tails, and option 3 if mismatch. Problem? Yes. The probability of mismatch is $\Pr[HT] + \Pr[TH] = 1/2$. Not $1/3–1/3–1/3$. 

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<td></td>
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</tr>
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Repeat the trial

We could flip two coins and say it’s option 1 if HH, option 2 if TT, option 3 if HT, and do over if TH. Problem? Maybe. There is an infinite number of outcomes...

Procedure selects option 1 if:

- HH on the first trial
- or TH on the first trial and HH on the second trial
- or TH on the first two trials and HH on the third trial
- ...

Pr(option 1)

\[
= \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} + \ldots
\]

\[
= \sum_{i=1}^{\infty} \frac{1}{4^i}
\]

\[
= \frac{1}{4} \times \sum_{i=0}^{\infty} \frac{1}{4^i}
\]

\[
= \frac{1}{4} \left( \frac{1}{1-\frac{1}{4}} \right)
\]

\[
= \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}.
\]
Aside: Geometric sum

\[ x = \sum_{i=0}^{\infty} p^i \]

the sum we want

\[ x = p^0 + p^1 + p^2 + p^3 + \ldots \]

expand

\[ px = p^1 + p^2 + p^3 + p^4 + \ldots \]

multiply by \( p \)

\[ p^0 + px = p^0 + p^1 + p^2 + p^3 + p^4 + \ldots \]

add \( p^0 \)

\[ p^0 + px = x \]

defn of \( x \)

\[ p^0 = x - px \]

subtract \( px \)

\[ 1 = x(1 - p) \]

factor/simplify

\[ \frac{1}{1-p} = x \]

divide by \( 1 - p \)
Infinite sample space

\[ S = \{ HH, HT, TT, TH : HH, TH : HT, TH : TT, TH : TH : HH, TH : TH : HT, TH : TH : TT, \ldots \} \]
\[ = \{ (TH)^n : HH, (TH)^n : HT, (TH)^n : TT | n \in \mathbb{N} \} \]

The probability space is:

\[ \Pr((TH)^n : HH) = \Pr((TH)^n : HT) = \Pr((TH)^n : TT) = \frac{1}{4^{n+1}}. \]

Note: \[ \sum_{n=0}^{\infty} 3 \times \frac{1}{4^{n+1}} = \frac{3}{4} \times \frac{1}{1-1/4} = \frac{3}{4} \times \frac{4}{3} = 1. \]

Non-negative and sums to one, valid probability space!