Binomial Theorem, Inclusion/Exclusion

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Overview

What's coming up

- We've just started talking about *combinatorics*, the art of counting things.
  - Our move: get good at counting certain kinds of things, like sequences. Then count other things by finding bijections to sets of sequences.
- $\binom{n}{k}$: the number of ways to pick k items from a set of n options.
- Leads into *probability*.
- And finally, graphs and trees: canonical "discrete structures."

k bits out of n

How many n-bit sequences contain exactly k ones? We talked about the bijection between subsets of an n-element set and n-bit sequences.

Example: A 4-element subset of $\{x_1, x_2, \ldots, x_8\}$ and the associated 8-bit sequence:

$$\begin{array}{cccccccccc} \{ & & x_2, & & x_4, & & & x_7, & x_8 & \} \\ (& 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 &). \end{array}$$

By the subset rule, the number of n-bit sequences with exactly k 1s is $\binom{n}{k}$.

Counting donuts

If there are three varieties of donuts at the bakery, how many different ways can you create a box of six?

- 6 Boston cream, 0 coconut, 0 glazed
- 4 Boston cream, 1 coconut, 1 glazed
- 2 Boston cream, 3 coconut, 1 glazed
- …

Counting donuts

$A =$ all ways to select a half-dozen doughnuts when three varieties are available

$B =$ all 8-bit sequences with exactly two 1s

$$\underbrace{0\ 0\ 0\ 0}_{\text{Boston cream}} \quad \underbrace{\qquad}_{\text{coconut}} \quad \underbrace{0\ 0}_{\text{glazed}}$$

Put our 6 donuts into the three bins. Note: Every way we can choose donuts becomes a pattern. And, every pattern (with 6 donuts) corresponds to a valid choice.

Use 1s to mark the gaps.

$$\underbrace{0\ 0\ 0\ 0}_{\text{Boston cream}} \quad 1 \quad \underbrace{\qquad}_{\text{coconut}} \quad 1 \quad \underbrace{0\ 0}_{\text{glazed}}$$ Now, squeeze: 00001100.

Every 8-bit pattern with exactly 2 ones is a valid donut order. Every valid donut order can be encoded with an 8-bit pattern with exactly 2 ones. Same size!

Time to solve the donuts!

$A =$ all ways to select a half-dozen doughnuts when three varieties are available

$B =$ all 8-bit sequences with exactly two 1s

Now, we know that $|B| = \binom{8}{2} = 28$, so 28 ways to select a half-dozen donuts when three varieties available!

Donuts and separators rule

Rule: All ways to select n donuts when k varieties are available:

$$\binom{n+k-1}{k-1}$$

There's $k-1$ separators (1 bits) and n donuts (0 bits). Need to put the $k-1$ separators into the $n+k-1$ slots.

Binomials to powers: Examples

$$
\begin{aligned}
(a+b)^2 &= aa + ab + ba + bb \\
&= a^2 + 2ab + b^2
\end{aligned}
$$

$$
\begin{aligned}
(a+b)^3 &= aaa + aab + aba + abb \\
&\quad + baa + bab + bba + bbb \\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}
$$

$$
\begin{aligned}
(a+b)^4 &= aaaa + aaab + aaba + aabb \\
&\quad + abaa + abab + abba + abbb \\
&\quad + baaa + baab + baba + babb \\
&\quad + bbaa + bbab + bbba + bbbb \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}
$$

How about $(a+b)^n$? How many terms consist of exactly k bs? Since it's all combinations of an a and b in each position, there are $\binom{n}{k}$ such terms.

Binomial theorem

Theorem: For all $n \in \mathbb{N}, a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Sometimes $\binom{n}{k}$ called the *binomial coefficient* because of this connection.

Pets and sets

S: Set of all students in CS0220.

$D \subseteq S$: Set of all students in CS0220 who have a pet dog.

$C \subseteq S$: Set of all students in CS0220 who have a pet cat.

$D \cup C$: Set of all students in CS0220 who have a pet dog *or* cat.

$|D \cup C| = |D| + |C|$? Handles people who have neither correctly. Handles people who have one kind of pet correctly. Messes up on people who have both.

Formulas for union

What's wrong with each formula for $|C \cup D|$?

- $|C| + |D|$? Double counted people who have both.
- $|C \setminus D| + |D \setminus C|$? Skipped people who have both.
- $|C \setminus D| + |D \setminus C| + |C \cap D|$? Actually, that should work. But, set difference can be tricky.
- $|C| + |D| - |C \cap D|$? Nailed it. Correct for double counting

Inclusion-Exclusion rule for two sets

Rule: For two sets S_1 and S_2,

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|.$$

Example:

- $S_1 = \{ \text{Jania, Josh} \}$: HTAs with a *j* in their name.
- $S_2 = \{ \text{Ben, Jania} \}$: HTAs with an *n* in their name.
- $S_1 \cap S_2 = \{ \text{Jania} \}$: HTAs with both a *j* and an *n* in their name.
- $S_1 \cup S_2 = \{ \text{Jania, Josh, Ben} \}$: HTAs with either a *j* or an *n* in their name.
- $|\{ \text{Jania, Josh, Ben} \}| = |\{ \text{Jania, Josh} \}| + |\{ \text{Ben, Jania} \}| - |\{ \text{Jania} \}|$

Generalize to three sets

S: Set of all students in CS0220.

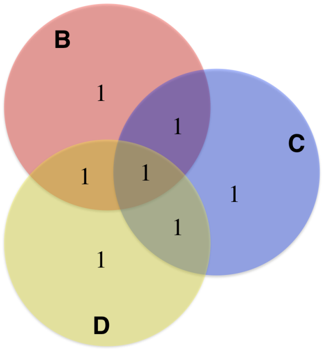
$D \subseteq S$: Set of all students in CS0220 who have a pet dog.

$C \subseteq S$: Set of all students in CS0220 who have a pet cat.

$B \subseteq S$: Set of all students in CS0220 who have a pet bunny.

How express $|B \cup C \cup D|$ in terms of size of *intersections* of sets?

Visual analysis



$$|B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D|$$

Inclusion-Exclusion rule for three sets

Rule: For three sets S_1, S_2, S_3,

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| = \quad & |S_1| + |S_2| + |S_3| \\ & -|S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ & +|S_1 \cap S_2 \cap S_3|. \end{aligned}$$

Example:

- $S_1 = \{ \text{Jania, Allie} \}$: TAs with an *a* in their name.
- $S_2 = \{ \text{Allie, Ben, Tyler} \}$: TAs with an *e* in their name.
- $S_3 = \{ \text{Jania, Allie} \}$: TAs with an *i* in their name.
- $S_1 \cap S_2 \cap S_3 = \{ \text{Allie} \}$: TAs with an *a* and an *e* and an *i* in their name.
- $|\{ \text{Jania, Allie, Ben, Tyler} \}| = |\{ \text{Jania, Allie} \}| +$ $|\{ \text{Allie, Ben, Tyler} \}| + |\{ \text{Jania, Allie} \}| - |\{ \text{Allie} \}|$ $-|\{ \text{Jania, Allie} \}| - |\{ \text{Allie} \}| + |\{ \text{Allie} \}|$

Sets of permutations

In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- $(4, 6, 5, 0, 1, 8, 3, 2, 9, 7)$ nope.
- $(0, 4, 6, 1, 8, 5, 9, 3, 7, 2)$ 04!
- $(3, 4, 2, 0, 5, 6, 1, 9, 8, 7)$ 42!
- $(3, 9, 4, 1, 2, 7, 0, 5, 6, 8)$ nope.
- $(0, 2, 6, 3, 7, 8, 4, 9, 5, 1)$ nope.

P_{60}: permutations of 0 through 9 that contain 60.

P_{04}: permutations of 0 through 9 that contain 04.

P_{42}: permutations of 0 through 9 that contain 42.

Want: $|P_{60} \cup P_{04} \cup P_{42}|$.

Inclusion-exclusion, constrained permutation

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60}| = ?$

Clever trick: In P_{60}, can view "60" as a unit. So, each element of P_{60} is a permutation of $\{1, 2, 3, 4, 5, 7, 8, 9, 60\}$. Therefore, $|P_{60}| = 9!$. $|P_{04}| = 9!$. $|P_{42}| = 9!$.

Pairwise intersections

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}| \\
&= 3 \times 9! \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60} \cap P_{04}| =$? Trick works again! Can view “604” as a unit. So, each element is a permutation of $\{1, 2, 3, 5, 7, 8, 9, 604\}$. Therefore, 8!.

$|P_{42} \cap P_{04}| =$? Trick works again! Can view “042” as a unit. So, 8!.

$|P_{60} \cap P_{42}| =$? Trick fails! Wait, no, just changes. Now, each element is a permutation of $\{1, 3, 5, 7, 8, 9, 60, 42\}$. Still 8!.

Three-way intersection

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad + |P_{60} \cap P_{04} \cap P_{42}| \\
&= 3 \times 9! - 3 \times 8! \\
&\quad + |P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60} \cap P_{04} \cap P_{42}| =?$. Yay, trick works again! Can view "6042" as a unit. So, each element is a permutation of $\{1, 3, 5, 7, 8, 9, 6042\}$. Therefore, 7!.

$|P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720.$

n-way Inclusion-Exclusion

$|S_1 \cup S_2 \cup \cdots \cup S_n| =$

	the sum of the sizes of the individual sets
minus	the sizes of all two-way intersections
plus	the sizes of all three-way intersections
minus	the sizes of all four-way intersections
plus	the sizes of all five-way intersections, etc.

Hyper-mathy version:

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{X \in \mathcal{P}([1,n]) - \emptyset} (-1)^{|X|+1} \left|\bigcap_{i \in X} S_i\right|$$