Induction Continued, Strong Induction

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Overview

1. Induction from a Starting Point

2. Strong Induction (5.2)

3. Optional topics
   - Stacking game
   - More on strong induction
Induction starting at a value other than 0

Our previous recipe for induction: to show $P(n)$ for every $n \in \mathbb{N}$, it suffices to show

- $P(0)$
- For an arbitrary $n$, if $P(n)$ holds, then $P(n + 1)$ holds.

Claim: to show $Q(n)$ for every $n \geq k$, it suffices to show

- $Q(k)$
- for an arbitrary $n \geq k$, if $Q(n)$ holds, then $Q(n + 1)$ holds.

Proof. Suppose we have a predicate $Q(n)$ with those properties. Define $P(n) = Q(n + k)$. We show $\forall n, P(n)$ holds by induction. $P(0) = Q(0 + k)$ holds, by assumption 1. Suppose $P(n)$ holds for an arbitrary $n$. This means $Q(n + k)$ holds. Since $n + k \geq k$, assumption 2 says $Q(n + k + 1)$ holds. And this is $P(n + 1)$. So by induction, we have shown $P(n) = Q(n + k)$ holds for all $n$: that is, $Q(n)$ holds for all $n \geq k$. 
Example

**Theorem.** For every natural number $n \geq 5$, $2^n > n^2$.

**Proof.** By induction on $n$ starting at 5, with predicate $P(n)$ defined to be $2^n > n^2$.

Base case: $2^5 = 32 > 25 = 5^2$.

Inductive step: suppose $n \geq 5$ and $2^n > n^2$. We want to show $2^{n+1} > (n + 1)^2$. Since $n \geq 5$, we have $2n + 1 \leq 3n \leq n^2$. So:

\[
(n + 1)^2 = n^2 + 2n + 1 \\
\leq n^2 + n^2 \\
< 2^n + 2^n \\
= 2^{n+1}
\]

This concludes our proof by induction from a starting point.
Strong Induction

Compare these two different “induction step” rules:

*Ordinary induction:* for all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$.

*Strong induction:* for all $n \in \mathbb{N}$, $P(0), P(1), \ldots, P(n)$ together imply $P(n + 1)$.

Strong induction lets you assume more in your inductive step. If you plan to use it in a proof, make sure you say so up front.
A recipe

To prove by strong induction that $P(n)$ holds for every $n \geq k$, we need to show that

- $P(k)$ holds.
- For any $n \geq k$, if $P(k), \ldots, P(n)$ all hold, then $P(n + 1)$ holds.

Combines strong induction and induction from a starting point! What happens when $k = 0$?
Another version of strong induction

We can phrase the strong induction principle differently:

To show $P(n)$ holds for all $n \in \mathbb{N}$, it is enough to show that for an arbitrary $n$, if $P(k)$ holds for every $k < n$, then $P(n)$ holds.

No base case! Why not?
Example usage of strong induction

**Theorem:** Every integer greater than 1 is a product of primes.

Our proof will use strong induction on $P(n)$ “$n$ is a product of primes”.

We need to prove that $P(n)$ holds for all $n \geq 2$.

**Base Case** ($n = 2$): $P(2)$ is true because 2 is prime, so it is the product of (one) primes.
Inductive step

**Inductive step**: Suppose that \( n \geq 2 \) and that \( k \) is a product of primes for every integer \( k \) where \( 2 \leq k \leq n \). We must show that \( P(n + 1) \), that is, \( n + 1 \) is a product of primes. We proceed by cases:

- \( n + 1 \) is prime: It is the product of (one) primes, so \( P(n + 1) \) holds.
- \( n + 1 \) is not prime: By definition, \( n + 1 = km \) for some integers \( k, m \) such that \( 2 \leq k, m \leq n \). By the strong induction hypothesis, both \( k \) and \( m \) are products of primes. So, therefore, so is \( km \) and so is \( n + 1 \). Again, \( P(n + 1) \) holds.

That completes the cases. And, that completes the strong induction proof.
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: (0, 1).
- 4? No.
- 5? Yes, one 5-unit coin: (1, 0).
- 6? Yes, two 3-unit coins: (0, 2).
- 7? No.
- 8? Yes: (1, 1).
- 9? Yes: (0, 3).
- 10? Yes: (2, 0).
- 11? Yes: (1, 2).
Coin Theorem

**Theorem:** You can make all values 8 or larger using coins of value 3 and 5.

**Proof:** We proceed by strong induction with induction hypothesis \( P(n) \): There is a collection of coins whose value is \( n \).

**Base case:** \( P(8) \) is true because we can make 8 using \((1, 1)\).

**Inductive step:** Assume \( P(k) \) holds for all integers \( 8 \leq k \leq n \), and now prove \( P(n + 1) \) holds. We argue by cases:

- \( n = 9 \): \((0, 3)\)
- \( n = 10 \): \((2, 0)\)
- \( n \geq 11 \): By strong induction, we can make the value \( n - 2 \), then add a 3-unit coin to get \( n + 1 \).

That completes the proof by strong induction, which proves the theorem. QED.
Every number is interesting

**Theorem.** Every natural number is interesting.

**Proof.** We proceed by strong induction with the property $P(n) = “n$ is interesting.”

Base case: 0 is certainly interesting!

Inductive case: suppose 0, . . . , $n$ are all interesting. If $n + 1$ were not interesting, it would be the smallest noninteresting number. But that’s interesting!
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two positive integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

| 7 | 5 2 | 10 |
| 2 3 2 | 16 |
| Example: 2 2 1 2 | 18 |
| 1 2 1 2 | 19 |
| 1 2 1 1 1 | 20 |
| 1 1 1 1 1 1 1 | 21 |
Irrelevance theorem

**Theorem:** No matter how you play the game starting from \( n \geq 1 \), your score will be \( n(n-1)/2 \).

**Proof:** The proof is by strong induction with \( P(n) \) as the proposition that every way of playing starting with the number \( n \) gives a score of \( n(n-1)/2 \).

**Base case:** Starting from a list of \( n = 1 \), the game ends immediately with a score of \( n(n-1)/2 = 1(0)/2 = 0 \). \( P(1) \) is true.

**Inductive step:** We must show that \( P(1), \ldots, P(n) \) imply \( P(n+1) \) for all \( n \geq 1 \). Assume that \( P(1), \ldots, P(n) \) are all true and we are playing starting with \( n + 1 \). The first move must break \( n + 1 \) into positive numbers \( a \) and \( b \) with \( a + b = n + 1 \). The total score for the game is the sum of points for this first move plus the scores obtained from playing with \( a \) and \( b \):
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
+ \text{ score from starting with } a \\
+ \text{ score from starting with } b \quad \text{ rules of game}
\]

\[
= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \quad \text{ inductive hyp.}
\]

\[
= 2ab + a^2 - a + b^2 - b
\]

\[
= (a+b)^2 - (a+b) \quad \text{ algebra}
\]

\[
= (n+1)^2 - (n+1) \quad \text{ Defn } a, b
\]

\[
= (n+1)n/2 \quad \text{ algebra}
\]

We showed \(P(1), P(2), \ldots, P(n)\) implies \(P(n + 1)\), showing the claim true by strong induction. QED.
We can prove that this principle of strong induction works, using regular induction. Suppose: for all $n \in \mathbb{N}$, if $P(k)$ holds for every $k < n$, then $P(n)$ holds. (Call this hypothesis $H$.)

We’ll proceed by induction on the property $Q(n) = “P(k) \text{ holds for every } k < n.”$

Base case: $Q(0)$ holds, since $P(k)$ holds for every $k < 0$.

Inductive step: Suppose $Q(n)$ holds. So $P(k)$ holds for every $k < n$. By our hypothesis $H$, $P(n)$ holds. Thus, $P(k)$ holds for every $k < n + 1$, which is $Q(n + 1)$. 

Proving strong induction
Strong Induction vs. Induction vs. Well Ordering (5.3)

Covered in the book, Chapter 2.

**Idea:** If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn’t the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we’re showing that being true up to some value \( n \) means it can’t be false for \( n \).

And, strong induction is just ordinary induction with a universal quantifier in its key proposition.

So, three different names for the same thing. Use is a matter of taste.