Functions, Injectivity, Surjectivity, Bijections

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Overview

1. Relation Diagrams (4.4.1)

2. Relational Images (4.4.2)
Binary relations

Definition. A *binary relation*, $R$, consists of a set, $A$, called the **domain** of $R$, a set, $B$, called the **codomain** of $R$, and a subset of $A \times B$ called the **graph** of $R$. 
Properties of relations

A binary relation:

- is a *partial function* when it has the $\leq 1$ arrow out] property.
  
  Book: “function”. Us: “function” is $\leq 1$ arrow out] property.

- is *surjective* when it has the $\geq 1$ arrows in] property.

- is *total* when it has the $\geq 1$ arrows out] property.

- is *injective* when it has the $\leq 1$ arrow in] property.

- is *bijective* when it has both the $\leq 1$ arrow out] and the $\leq 1$ arrow in] properties.
Example relation #1

partial function: $\leq 1$ out. surjective: $\geq 1$ in. total: $\geq 1$ out. injective: $\leq 1$ in. bijective: $= 1$ out and $= 1$ in.

Partial function; surjective; total. Not injective, not bijective. Summary: a surjective function. (Implies partial function and total.)
Example relation #2

Partial function: \([\leq 1 \text{ out}].\) Surjective: \([\geq 1 \text{ in}].\) Total: \([\geq 1 \text{ out}].\) Injective: \([\leq 1 \text{ in}].\) Bijective: \([= 1 \text{ out}]\) and \([= 1 \text{ in}].\)

Partial function; total; injective. Not surjective, not bijective.
Summary: an injective function. (Implies partial function and total.)
Example relation #3

partial function: \( \leq 1 \) out. surjective: \( \geq 1 \) in. total: \( \geq 1 \) out. injective: \( \leq 1 \) in. bijective: \( = 1 \) out and \( = 1 \) in.

Equation \( y = \frac{1}{x^2} \) on \( \mathbb{R}^+ \). \( x \) is an element in the domain, \( y \) is an element in the co-domain.

Partial function; surjective; total; injective; bijective.
Summary: a bijective (partial) function. (Implies everything else.)
Example relation #4

partial function: $\leq 1 \text{ out}$. surjective: $\geq 1 \text{ in}$. total: $\geq 1 \text{ out}$. injective: $\leq 1 \text{ in}$. bijective: $= 1 \text{ out}$ and $= 1 \text{ in}$.

Equation $y = \frac{1}{x^2}$ on $\mathbb{R}$.

Partial function. Not anything else.
Image definition

Definition. The *image* of a set $Y \subseteq A$ under a relation $R : A \rightarrow B$, written $R(Y)$, is the subset of elements of the codomain $B$ of $R$ that are related to some element in $Y$.

In terms of the relation diagram, $R(Y)$ is the set of points with an arrow coming in that starts from some point in $Y$.

$$R(Y) = \{ x \in B \mid \exists y \in Y, y R x \}.$$
Inverse definition

Definition: The *inverse* $R^{-1}$ of a relation $R : A \to B$ is the relation from $B$ to $A$ defined by the rule

$$b R^{-1} a \iff a R b.$$ 

Definition: The image of a set under the relation $R^{-1}$ is called the *inverse image* of the set. That is, the inverse image of a set $X$ under the relation $R$ is defined to be $R^{-1}(X)$.

Example: $x R y$ iff there’s a dictionary word with first letter $x$ and second letter $y$. The image $R(\{c, k\})$ is the letters that can appear after ‘c’ or ‘k’ at the beginning of a word. It’s the set $\{a, e, h, i, l, n, o, r, u, v, w, y, z\}$.

The inverse image $R^{-1}(\{c, k\})$ is the letters that can appear before ‘c’ or ‘k’ at the beginning of a word. It’s the set $\{a, e, i, o, s, t, u\}$. 

Inverses of relations

What can we infer about $R^{-1}$ if $R$ is:

- partial function? injective
- surjective? total
- total? surjective
- injective? partial function
- bijective? bijective
- function? injective and surjective
More examples to consider

Can you come up with examples of relations on $\mathbb{R}$ that are:

- Surjective, not a partial function?
- A partial function, total, injective but not surjective?
- Everything (a bijective function)? (Something different from $y = x$!)