The Language of Set Theory

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CS 0220 2023

February 8, 2023
Overview

1. Proof rules for quantifiers
2. Sets Definitions (4.1–4.1.1)
3. Sets Operations (4.1.2–4.1.5)
forall proof rules

Introduction: To **prove** a forall goal $\forall x : T, G(x)$:
Suppose you have a (new, freshly named) $x : T$ in your context, and prove $G(x)$ for that new $x$.

I want to show that every number is either prime or the product of two other numbers. Suppose $n$ is a number. Show that $n$ is prime or $n$ is the product of two other numbers.

Elimination: To **use** a forall hypothesis $\forall x : T, H(x)$:
If $t : T$ is any term of the right type, then you can add a hypothesis $H(t)$.

I know that every number is either prime or the product of two other numbers. Therefore, I know that either 2 is prime or 2 is the product of two other numbers. I know that either 5 is prime or 5 is the product of two other numbers...
Exists proof rules

To **prove** an existential goal \( \exists x : T, G(x) \):

Provide a witness.

I want to show that there is a perfect square whose final digit is 4. I claim that my witness is 64. Then, I must show that 64 is a perfect square and the final digit of 64 is 4.

Elimination: To **use** an existential hypothesis \( \exists x : T, H(x) \):
you can create a (new, freshly named) \( t : T \), and add a hypothesis \( H(t) \). "Give a name to the witness."

I know that there is a perfect square whose final digit is 4. Let’s call this perfect square \( ps \). I know that \( ps \) is a perfect square and the final digit of \( ps \) is 4.
Mathematical languages

We’re building up a *formal language* for talking about propositions.

Natural language is confusing and ambiguous. Ours is not. (Fingers crossed!)

A new part of our language today: *sets*. Are we *defining* sets? Or introducing them as an *atomic concept*?

Either way! Really useful *vocabulary* for talking about things, mathematical and otherwise.
Set Definition

Definition (informal): A set is a bunch/collection/group of objects.

Definition: The elements of the set are the objects contained in that set.

Sets can contain numbers, ordered sequences of numbers, strings, names, or other sets.

Objects are either in the set or not in the set. We don’t have a concept of an object being in a set multiple times. It’s a Boolean property.

We write curly braces around a comma-separated list to build a set.

Examples:

- \( H = \{ \text{Ben, Jania, Josh, Tyler} \} \)
- \( I = \{ \text{this laptop, this slide clicker, that projector screen} \} \)
- \( J = \{ \text{“this laptop”, “this slide clicker”, “that projector screen”} \} \)
- \( \mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \} \)
Elements

- $H = \{ \text{Ben, Jania, Josh, Tyler} \}$
- $I = \{ \text{this laptop, this slide clicker, that projector screen} \}$
- $\mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \}$

Definition: We say $x \in S$ if $x$ is an element of or in or a member of the set $S$.

- Ben $\in H$? Yes.
- this laptop $\in H$? No. This laptop $\notin H$.
- this laptop $\in I$? Yes.
- Jania $\in \mathbb{N}$? No. Jania $\notin \mathbb{N}$. 
Sets of sets

- $A = \{1, 4, 9\}$
- $B = \{\{1, \{4\}\}, \{9\}\}$

- $1 \in A$?
  - Yes.

- $1 \in B$?
  - No, but $\{1, \{4\}\} \in B$.

- $\exists x: \mathbb{N}, x \in B \land 1 \in x$?
  - Yes, $x = \{1, \{4\}\} \in B$ and $1 \in x$. 
Some Sets of Numbers

- $\emptyset = \{\}$ (empty set, null set)
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
- $\mathbb{Q} = \{1/2, -4/15, 21, \ldots\}$ (rationals)
- $\mathbb{R} = \{\sqrt{2}, -\pi, 21, \ldots\}$ (real numbers)
- $\mathbb{C} = \{i/2, 15 - i, \sqrt{7}, 21, \ldots\}$ (complex numbers)

Superscript plus limits to (strictly!) positive values: $\mathbb{Z}^+ = \mathbb{N}^+$.

Superscript minus limits to negative values: $21 \notin \mathbb{R}^-$. 

∅ = {} (empty set, null set)

N = {0, 1, 2, 3, 4, …} (non-negative integers)

Z = {…, –3, –2, –1, 0, 1, 2, 3, …} (integers)

Q = {1/2, –4/15, 21, …} (rationals)

R = {√2, –π, 21, …} (real numbers)

C = {i/2, 15 – i, √7, 21, …} (complex numbers)

Superscript plus limits to (strictly!) positive values: Z⁺ = N⁺.

Superscript minus limits to negative values: 21 ∉ R⁻.
Subsets

Definition: One set is a *subset* of another if every element of the first set is also an element of the second.

We write \( S \subseteq T \) to say the set \( S \) is a subset of set \( T \). So, \( S \subseteq T \) means \( \forall x \in S, x \in T \). Could also write \( \forall x, x \in S \rightarrow x \in T \).

Examples:

- \( \mathbb{N} \subseteq \mathbb{Z} \)? Yes, every positive integer is also an integer.
- \( \mathbb{Z}^+ \subseteq \mathbb{N} \)? Yes, every positive integer is also a non-negative integer.
- \( \mathbb{C} \subseteq \mathbb{Z} \)? No, \( \mathbb{C} \nsubseteq \mathbb{Z} \). Some (many!) complex numbers are not integers. Although, \( \mathbb{Z} \subseteq \mathbb{C} \).
- \( \mathbb{N} \subseteq \mathbb{N} \). Yes, if sets are equal, all of the first must also be in the second!

Note: \( \{1, 2, 3\} \subseteq \{1, 2, 3, 4\} \) looks a little bit like \( 3 \leq 4 \). We write \( A \subset B \) to rule out equality (like \( a < b \)).
Operations on sets: Union

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *union* of sets $X$ and $Y$, $X \cup Y$, consists of every element that is in either $X$ or $Y$. In other words, $z \in X \cup Y$ means $z \in X \lor z \in Y$.

Example: $A \cup B = \{b, e, n, j, a, i\}$. (Order doesn’t matter: = $\{a, b, e, i, j, n\}$)
Operations on sets: Intersection

- \( A = \{b, e, n\} \)
- \( B = \{j, a, n, i\} \)
- \( C = \{j, o, s, h\} \)
- \( D = \{t, y, l, e, r\} \)

Definition: The *intersection* of sets \( X \) and \( Y \), \( X \cap Y \), consists of every element that is in both \( X \) and \( Y \). In other words, \( z \in X \cap Y \) means \( z \in X \land z \in Y \).

Example: \( B \cap C = \{j\} \). \( B \cap D = \emptyset \)
Operations on sets: Set difference

- \( A = \{b, e, n\} \)
- \( B = \{j, a, n, i\} \)
- \( C = \{j, o, s, h\} \)
- \( D = \{t, y, l, e, r\} \)

Definition: The set difference of sets \( X \) and \( Y \), \( X \setminus Y \), consists of every element that is in \( X \) but not in \( Y \). In other words, \( z \in X \setminus Y \) means \( z \in X \land z \notin Y \).

Example: \( A \setminus D = \{b, n\} \).

Example: \( D \setminus A = \{t, y, l, r\} \).
Operations on sets: Symmetric difference

- \( A = \{b, e, n\} \)
- \( B = \{j, a, n, i\} \)
- \( C = \{j, o, s, h\} \)
- \( D = \{t, y, l, e, r\} \)

Definition: The *symmetric difference* of sets \( X \) and \( Y \), \( X \triangle Y \), consists of every element that is in \( X \) but not in \( Y \) or in \( Y \) but not \( X \). In other words, \( z \in X \triangle Y \) means 
\[(z \in X \land z \notin Y) \lor (z \in Y \land z \notin X).\]
That is, \( z \in X \) XOR \( z \in Y \).

Example: \( C \triangle B = \{o, s, h, a, n, i\} \).

Example: \( A \triangle D = \{b, l, n, r, t, y\} \). (Remember: order doesn’t matter)
Operations on sets: Complement

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The complement of a set $X$, $\bar{X}$, is defined with respect to some universe of possible elements $U$. It consists of every possible element that is not in $X$. In other words, $\bar{X} = U \setminus X$.

Example: If $U$ is the universe of all letters in English, $\bar{A} = \{a, cd, f, g, h, i, j, k, l, m, o, p, q, r, s, t, u, v, w, x, y, z\}$.

Example: If $U = \mathbb{Z}$, $\mathbb{Z}^- = \mathbb{Z}^+ \setminus \{0\}$.
Disjoint sets

Definition: Sets $X$ and $Y$ are *disjoint* if they have no elements in common.

$X \cap Y = \emptyset$ or

$X \subseteq Y$. 
Operations on sets: Power set

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The power set of a set $X$, $\mathcal{P}(X)$, is the set of all subsets of $X$. In other words, $\forall x \in \mathcal{P}(X), x \subseteq X$ and $\forall x \subseteq X, x \in \mathcal{P}(X)$.

Example: $\mathcal{P}(A) = \{\{\}, \{b\}, \{e\}, \{n\}, \{b, e\}, \{b, n\}, \{e, n\}, \{b, e, n\}\}$.

Example: $\mathcal{P}(C) = \{\{\}, \{j\}, \{o\}, \{s\}, \{h\}, \{j, o\}, \{j, s\}, \{j, h\}, \{o, s\}, \ldots, \{j, o, s, h\}\}$.

Example: $\mathcal{P}(\emptyset) = \{\emptyset\}$. 
Operations on sets: Cardinality

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *cardinality* of a set $X$, $|X|$, is the count of the number of (unique) elements in $X$.

Example: $|A| = 3$, $|B| = 4$, $|C| = 4$, $|D| = 5$

Example: $|\emptyset| = 0$.

Example: If $|A| = n$, $|\mathcal{P}(A)| = 2^n$. Each subset consists of a decision of whether to include or not include (2 possibilities) each of the $n$ elements of $A$. 
Building sets with predicates

General form: \{ description of a set | filter on the set \}.

Examples:

- \( A = \{ n \in \mathbb{N} \mid n = 2k + 1 \text{ for some integer } k \} \)
- \( B = \{ x \in \mathbb{R} \mid x^2 > 1 \} \)

Note: Python has a notation for this idea.
Products of sets

- \( C = \{2, 5\} \)
- \( D = \{a, b, c\} \)

- \( C \times D = \{(2, a), (2, b), (2, c), (5, a), (5, b), (5, c)\} \)
- \( \mathbb{N} \times D = \{(0, a), (0, b), (0, c), (1, a), (1, b), \ldots\} \)
- \( \mathbb{N} \times \mathbb{N} = \text{the set of ordered pairs of natural numbers} \)

Ordered pair: \((2, 0)\) is not the same as \((0, 2)\)!