# The Language of Set Theory

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# Overview

# forall proof rules

Introduction: To **prove** a forall goal $\forall x : T, G(x)$:
Suppose you have a (new, freshly named) $x : T$ in your context, and prove $G(x)$ for that new $x$.

I want to show that every number is either prime or the product of two other numbers.
Suppose $n$ is a number. Show that $n$ is prime or $n$ is the product of two other numbers.

Elimination: To **use** a forall hypothesis $\forall x : T, H(x)$:
If $t : T$ is any term of the right type, then you can add a hypothesis $H(t)$.

I know that every number is either prime or the product of two other numbers.
Therefore, I know that either 2 prime or 2 is the product of two other numbers. I know that either 5 is prime or 5 is the product of two other numbers...

# Exists proof rules

To **prove** an existential goal $\exists x : T, G(x)$:
Provide a witness.

I want to show that there is a perfect square whose final digit is 4. I claim that my witness is 64. Then, I must show that 64 is a perfect square and the final digit of 64 is 4.

Elimination: To **use** an existential hypothesis $\exists x : T, H(x)$:
you can create a (new, freshly named) $t : T$, and add a hypothesis $H(t)$. "Give a name to the witness."

I know that there is a perfect square whose final digit is 4. Let's call this perfect square *ps*. I know that *ps* is a perfect square and the final digit of *ps* is 4.

# Mathematical languages

We're building up a *formal language* for talking about propositions.

Natural language is confusing and ambiguous. Ours is not. (Fingers crossed!)

A new part of our language today: *sets*. Are we *defining* sets? Or introducing them as an *atomic concept*?

Either way! Really useful *vocabulary* for talking about things, mathematical and otherwise.

# Set Definition

Definition (informal): A *set* is a bunch/collection/group of objects.

Definition: The *elements* of the set are the objects contained in that set.

Sets can contain numbers, ordered sequences of numbers, strings, names, or other sets.

Objects are either *in* the set or *not in* the set. We don't have a concept of an object being in a set multiple times. It's a Boolean property.

We write curly braces around a comma-separated list to build a set.

Examples:

- $H = \{$ Ben, Jania, Josh, Tyler $\}$
- $I = \{$ this laptop, this slide clicker, that projector screen $\}$
- $J = \{$ "this laptop", "this slide clicker", "that projector screen" $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

# Elements

- $H = \{$ Ben, Jania, Josh, Tyler $\}$
- $I = \{$ this laptop, this slide clicker, that projector screen $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

Definition: We say say $x \in S$ if $x$ is *an element of* or *in* or *a member of* the set $S$.

- Ben $\in H$?
- this laptop $\in H$?
- this laptop $\in I$?
- Jania $\in \mathbb{N}$?

- Yes.
- No. This laptop $\notin H$.
- Yes.
- No. Jania $\notin \mathbb{N}$.

# Sets of sets

- $A = \{1, 4, 9\}$
- $B = \{\{1, \{4\}\}, \{9\}\}$

- $1 \in A$?
- $1 \in B$?
- $\exists x : \mathbb{N}, x \in B \wedge 1 \in x$?

- Yes.
- No, but $\{1, \{4\}\} \in B$.
- Yes, $x = \{1, \{4\}\} \in B$ and $1 \in x$.

# Some Sets of Numbers

- $\emptyset = \{\}$ (empty set, null set)
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
- $\mathbb{Q} = \{1/2, -4/15, 21, \ldots\}$ (rationals)
- $\mathbb{R} = \{\sqrt{2}, -\pi, 21, \ldots\}$ (real numbers)
- $\mathbb{C} = \{i/2, 15 - i, \sqrt{7}, 21, \ldots\}$ (complex numbers)

Superscript plus limits to (strictly!) positive values: $\mathbb{Z}^+ = \mathbb{N}^+$.

Superscript minus limits to negative values: $21 \notin \mathbb{R}^-$.

# Subsets

Definition: One set is a *subset* of another if every element of the first set is also an element of the second.

We write $S \subseteq T$ to say the set $S$ is a subset of set $T$. So, $S \subseteq T$ means $\forall x \in S, x \in T$. Could also write $\forall x, x \in S \rightarrow x \in T$.

Examples:

- $\mathbb{N} \subseteq \mathbb{Z}$? Yes, every positive integer is also an integer.
- $\mathbb{Z}^+ \subseteq \mathbb{N}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{C} \subseteq \mathbb{Z}$? No, $\mathbb{C} \not\subseteq \mathbb{Z}$. Some (many!) complex numbers are not integers. Although, $\mathbb{Z} \subseteq \mathbb{C}$.
- $\mathbb{N} \subseteq \mathbb{N}$. Yes, if sets are equal, all of the first must also be in the second!

Note: $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ looks a little bit like $3 \leq 4$.
We write $A \subset B$ to rule out equality (like $a < b$).

# Operations on sets: Union

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *union* of sets $X$ and $Y$, $X \cup Y$, consists of every element that is in either $X$ or $Y$. In other words, $z \in X \cup Y$ means $z \in X \vee z \in Y$.

Example: $A \cup B = \{b, e, n, j, a, i\}$. (Order doesn't matter: = $\{a, b, e, i, j, n\}$)

# Operations on sets: Intersection

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *intersection* of sets $X$ and $Y$, $X \cap Y$, consists of every element that is in both $X$ and $Y$. In other words, $z \in X \cap Y$ means $z \in X \wedge z \in Y$.

Example: $B \cap C = \{j\}$. $B \cap D = \emptyset$

# Operations on sets: Set difference

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *set difference* of sets $X$ and $Y$, $X \setminus Y$, consists of every element that is in $X$ but not in $Y$. In other words, $z \in X \setminus Y$ means $z \in X \wedge z \notin Y$.

Example: $A \setminus D = \{b, n\}$.

Example: $D \setminus A = \{t, y, l, r\}$.

# Operations on sets: Symmetric difference

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *symmetric difference* of sets $X$ and $Y$, $X \triangle Y$, consists of every element that is in $X$ but not in $Y$ or in $Y$ but not $X$. In other words, $z \in X \triangle Y$ means $(z \in X \wedge z \notin Y) \vee (z \in Y \wedge z \notin X)$. That is, $z \in X$ XOR $z \in Y$.

Example: $C \triangle B = \{o, s, h, a, n, i\}$.

Example: $A \triangle D = \{b, l, n, r, t, y\}$. (Remember: order doesn't matter)

# Operations on sets: Complement

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *complement* of a set $X$, $\overline{X}$, is defined with respect to some universe of possible elements $U$. It consists of every possible element that is not in $X$. In other words, $\overline{X} = U \setminus X$.

Example: If $U$ is the universe of all letters in English, $\overline{A} = \{a, cd, f, g, h, i, j, k, l, m, o, p, q, r, s, t, u, v, w, x, y, z\}$.

Example: If $U = \mathbb{Z}$, $\mathbb{Z}^- = \overline{\mathbb{Z}^+} \setminus \{0\}$.

# Disjoint sets

Definition: Sets $X$ and $Y$ are *disjoint* if they have no elements in common.

$X \cap Y = \emptyset$ or

$X \subseteq \overline{Y}$.

# Operations on sets: Power set

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *power set* of a set $X$, $\mathcal{P}(X)$, is the set of all subsets of $X$. In other words, $\forall x \in \mathcal{P}(X), x \subseteq X$ and $\forall x \subseteq X, x \in \mathcal{P}(X)$.

Example: $\mathcal{P}(A) = \{\{\}, \{b\}, \{e\}, \{n\}, \{b, e\}, \{b, n\}, \{e, n\}, \{b, e, n\}\}$.

Example: $\mathcal{P}(C) = \{\{\}, \{j\}, \{o\}, \{s\}, \{h\}, \{j, o\}, \{j, s\}, \{j, h\}, \{o, s\}, \ldots, \{j, o, s, h\}\}$.

Example: $\mathcal{P}(\emptyset) = \{\emptyset\}$.

# Operations on sets: Cardinality

- $A = \{b, e, n\}$
- $B = \{j, a, n, i\}$
- $C = \{j, o, s, h\}$
- $D = \{t, y, l, e, r\}$

Definition: The *cardinality* of a set $X$, $|X|$, is the count of the number of (unique) elements in $X$.

Example: $|A| = 3, |B| = 4, |C| = 4, |D| = 5$

Example: $|\emptyset| = 0$.

Example: If $|A| = n$, $|\mathcal{P}(A)| = 2^n$. Each subset consists of a decision of whether to include or not include (2 possibilities) each of the $n$ elements of $A$.

# Building sets with predicates

General form: { description of a set | filter on the set }.

Examples:

- $A = \{n \in \mathbb{N} \mid n = 2k + 1 \text{ for some integer } k\}$
- $B = \{x \in \mathbb{R} \mid x^2 > 1\}$

Note: Python has a notation for this idea.

# Products of sets

- $C = \{2, 5\}$
- $D = \{a, b, c\}$

- $C \times D = \{(2, a), (2, b), (2, c), (5, a), (5, b), (5, c)\}$
- $\mathbb{N} \times D = \{(0, a), (0, b), (0, c), (1, a), (1, b), \ldots\}$
- $\mathbb{N} \times \mathbb{N} =$ the set of *ordered pairs* of natural numbers

*Ordered pair*: $(2, 0)$ is not the same as $(0, 2)$!