# First-Order Logic

## Predicates and Quantifiers

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# Overview

# Disjunctive normal form

Definition: A formula in *disjunctive normal form* is an OR of terms, where each term is an AND of variables or negations of variables.

$(A \wedge B \wedge \neg C) \vee (\neg B \wedge C)$

$A \vee B \vee (A \wedge B \wedge \neg C)$

Not in DNF: $(A \wedge B) \vee \neg(B \wedge C)$

# Disjunctive normal form is universal

**Theorem**: For every formula, there is an equivalent formula written in DNF.

**Proof**: You can read the terms off of the truth table, turning each "true" row into a conjunction of literals.

| $A$ | $B$ | $C$ | value | | |
|---|---|---|---|---|---|
| **F** | **F** | **F** | **F** | | |
| **F** | **F** | **T** | **T** | $\leftarrow$ | $\neg A \wedge \neg B \wedge C$ |
| **F** | **T** | **F** | **F** | | |
| **F** | **T** | **T** | **F** | | |
| **T** | **F** | **F** | **F** | | |
| **T** | **F** | **T** | **T** | $\leftarrow$ | $A \wedge \neg B \wedge C$ |
| **T** | **T** | **F** | **T** | $\leftarrow$ | $A \wedge B \wedge \neg C$ |
| **T** | **T** | **T** | **F** | | |

$$(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)$$

# Properties of disjunctive normal form

How big could the disjunctive normal form get? Big!

Definition: If every variable appears exactly once in every term in a disjunctive normal form expression, then it is in *full disjunctive normal form*.

| Book | Wikipedia/me |
|---|---|
| disjunctive form | disjunctive normal form |
| disjunctive normal form | full disjunctive normal form |

Given a formula in DNF (disjunctive normal form), can we determine whether it is satisfiable? Valid? Satisfiability is easy—a single term tells us a satisfying assignment. Validity is not obvious—a given term might exclude an assignment, but perhaps another picks it up?

# Conjunctive normal form

Definition: A formula in *conjunctive normal form* is an AND of clauses, where each clause is an OR of variables or negations of variables.

$(\neg A \vee \neg B \vee C) \wedge (B \vee \neg C)$

$\neg A \wedge B \wedge (\neg A \vee C)$

Not an example: $\neg A \vee B \wedge (\neg A \vee C)$

# Conjunctive normal form is universal

**Theorem**: For every formula, there is an equivalent formula written in CNF.

**Proof**: Negate the truth table. Write in DNF. Negate formula via DeMorgan's law. QED.

| $A$ | $B$ | $C$ | value | negated | | |
|---|---|---|---|---|---|---|
| **F** | **F** | **F** | **T** | **F** | | |
| **F** | **F** | **T** | **F** | **T** | $\leftarrow$ | $\neg A \land \neg B \land C$ |
| **F** | **T** | **F** | **T** | **F** | | |
| **F** | **T** | **T** | **T** | **F** | | |
| **T** | **F** | **F** | **T** | **F** | | |
| **T** | **F** | **T** | **F** | **T** | $\leftarrow$ | $A \land \neg B \land C$ |
| **T** | **T** | **F** | **F** | **T** | $\leftarrow$ | $A \land B \land \neg C$ |
| **T** | **T** | **T** | **T** | **F** | | |

DNF for negated: $(\neg A \land \neg B \land C) \lor \ldots$

CNF: $(A \lor B \lor \neg C) \land \ldots$

# Properties of conjunctive normal form

How big could the conjunctive normal form get? Big.

Definition: If every variable appears in every clause in a conjunctive normal form expression, then it is in *full conjunctive normal form*.

| Book | Wikipedia/me |
|---|---|
| conjunctive form | conjunctive normal form |
| conjunctive normal form | full conjunctive normal form |

Given a formula in CNF (conjunctive normal form), can we determine whether it is satisfiable? Valid? Validity is easy now—a single clause throws out an assignment, so a single clause makes the formula not valid. Satisfiability is not so clear—each clause knocks out some assignments, but not clear if the set of clauses miss anything.

# First-order Logic

The language of *propositional* logic: atoms and connectives. Every formula is either an atom, or one or more formulas related by a connective. $p \wedge q \rightarrow r$

The language of *first-order* (or *predicate*) logic:

- Variables: $x$, $y$, $n$, ...
- Function symbols: $f(x)$, $plus(a, b)$, ... (sometimes with notation)
- Predicate symbols: $P(x)$, $R(x, y)$, Prime$(n)$, ... propositions with placeholders
- Quantifiers: $\forall$, $\exists$
- ... and the same old connectives as before

# Technical specification

A well-formed *term* in first-order logic is

- a variable ($x, y, n, \ldots$), or
- a function symbol applied to the correct number of terms ($f(x)$, $plus(x, y)$, ...), or
- a constant symbol ($0, 1, \emptyset, \ldots$)

Terms represent "things."

A well-formed *formula* in first-order logic is

- a predicate symbol applied to the correct number of terms ($R(x, y)$, $Prime(n)$, ...), or
- one or more formulas joined by a connective ($P(x) \land Q(y)$, $\neg R(x, y)$, ...), or
- a quantifier, followed by a variable, followed by a formula ($\forall x : \mathbb{N}, P(x) \land Q(x)$)

Formulas represent "statements." (Like propositions?)

# Concept Check

Let $=$ and $R$ be predicate symbols and $+$ and $f$ be function symbols.

Top Hat question: which of the following are well-formed formulas?

- $x = 0 \vee x = 1 \vee x = 2$
- $f(x) \wedge f(y)$
- $\forall x : \mathbb{Z}, x + 0$
- $\exists x : \mathbb{Z}, \forall y : \mathbb{Z}, R(f(x), f(y))$
- $\forall x \wedge y = 2$

# Translations

From day 1:

- There is a perfect square whose final digit is 4.
  $\exists x : \mathbb{N}, PS(x) \land (fd(x) = 4)$
- Every number is either prime or the product of two other numbers.
  $\forall n : \mathbb{N}, Prime(n) \lor \exists p\, q : \mathbb{N}, n = p \cdot q$
- Every number is either prime or the product of two *smaller* numbers.
  $\forall n : \mathbb{N}, Prime(n) \lor \exists p\, q : \mathbb{N}, (p < n) \land (q < n) \land (n = p \cdot q)$
- Every even integer greater than two is the sum of two primes.
  $\forall n : \mathbb{N}, Even(n) \land (n > 2) \to \exists p\, q : \mathbb{N}, Prime(p) \land Prime(q) \land (n = p + q)$

# Try a few yourself!

You can make up some predicate and function symbols, like $TD(n)$ for "has two digits".

On Top Hat:

- $313(x^3 + y^3) = z^3$ has no solution when $x, y, z \in \mathbb{Z}^+$.
- There is a two-digit perfect square whose final digit is 4.
- Every prime number greater than 2 is odd.

# forall proof rules

Introduction: To **prove** a forall goal $\forall x : T, G(x)$:
Suppose you have a (new, freshly named) $x : T$ in your context, and prove $G(x)$ for that new $x$.

I want to show that every number is either prime or the product of two other numbers.
Suppose $n$ is a number. Show that $n$ is prime or $n$ is the product of two other numbers.

Elimination: To **use** a forall hypothesis $\forall x : T, H(x)$:
If $t : T$ is any term of the right type, then you can add a hypothesis $H(t)$.

I know that every number is either prime or the product of two other numbers.
Therefore, I know that either 2 prime or 2 is the product of two other numbers. I know that either 5 is prime or 5 is the product of two other numbers...

# Exists proof rules

To **prove** an existential goal $\exists x : T, G(x)$:
Provide a witness.

I want to show that there is a perfect square whose final digit is 4. I claim that my witness is 64. Then, I must show that 64 is a perfect square and the final digit of 64 is 4.

Elimination: To **use** an existential hypothesis $\exists x : T, H(x)$:
you can create a (new, freshly named) $t : T$, and add a hypothesis $H(t)$. "Give a name to the witness."

I know that there is a perfect square whose final digit is 4. Let's call this perfect square *ps*. I know that *ps* is a perfect square and the final digit of *ps* is 4.