Normal Forms (CNF, DNF)

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Overview

Arguing either way

(A) If my lily is overwatered, then my lily's leaves droop.
$P \to Q$.

(B) If my lily's leaves do not droop, then my lily is not overwatered.
$\neg Q \to \neg P$.

Same? Let's check.

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
F	**F**	**T**	**T**	**T**	
F	**T**	**T**	**F**	**T**	
T	**F**	**F**	**T**	**F**	
T	**T**	**T**	**F**	**F**	

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F	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**
T	**T**	**T**	**F**	**F**	**T**

The *implication* (A) and its *contrapositive* (B) are equivalent.

Converse

(A) If my lily is overwatered, then my lily's leaves droop.
$P \to Q$.

(C) If my lily's leaves droop, then my lily is overwatered.
$Q \to P$.

Same? Let's check.

P	Q	$P \to Q$	$Q \to P$
F	**F**	**T**	
F	**T**	**T**	
T	**F**	**F**	
T	**T**	**T**	

Can’t argue both ways

(A) If my lily is overwatered, then my lily’s leaves droop.
$P \to Q$.

(C) If my lily’s leaves droop, then my lily is overwatered.
$Q \to P$.

Same? Let’s check.

P	Q	$P \to Q$	$Q \to P$
F	**F**	**T**	**T**
F	**T**	**T**	**F**
T	**F**	**F**	**T**
T	**T**	**T**	**T**

The implication (A) and its *converse* (C) are not equivalent.

DeMorgan's Law

These two statements are equivalent:

- $\neg(P \wedge Q)$
- $\neg P \vee \neg Q$

They are equivalent because they have exactly the same truth table. (Or, because we can *prove* $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$.) You can think of the rule as negation "distributing" over AND, negating the inputs and switching the AND to OR. It's a very general and very useful rule.

Equivalence and validity: Definitions

A formula can be thought of as a function mapping variable assignments to truth values. Each row of the truth table shows one input and its corresponding output.

Definition: Two formulas over the same set of variables are *equivalent* if they evaluate to the same truth value under every variable assignment.

Definition: A formula is *valid* if it is always true regardless of variable assignment.

Example: $P \lor \neg P$

P	$\neg P$	$P \lor \neg P$
F	**T**	**T**
T	**F**	**T**

Equivalence and validity

A formula is valid iff it is equivalent to **T**.

Two formulas α and β are equivalent iff $\alpha \leftrightarrow \beta$ is valid.

Example: Show "P" is equivalent to "$\neg\neg P$".

P	$\neg P$	$\neg\neg P$	$P \leftrightarrow \neg\neg P$
F			
T			

Equivalence and validity

A formula is valid iff it is equivalent to **T**.

Two formulas α and β are equivalent iff $\alpha \leftrightarrow \beta$ is valid.

Example: Show "P" is equivalent to "$\neg\neg P$".

P	$\neg P$	$\neg\neg P$	$P \leftrightarrow \neg\neg P$
F	**T**		
T	**F**		

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Example: Show "P" is equivalent to "$\neg\neg P$".

P	$\neg P$	$\neg\neg P$	$P \leftrightarrow \neg\neg P$
F	**T**	**F**	
T	**F**	**T**	

Equivalence and validity

A formula is *valid* iff it is equivalent to **T**.

Two formulas α and β are equivalent iff $\alpha \leftrightarrow \beta$ is valid.

Example: Show "P" is equivalent to "$\neg\neg P$".

P	$\neg P$	$\neg\neg P$	$P \leftrightarrow \neg\neg P$
F	**T**	**F**	**T**
T	**F**	**T**	**T**

DeMorgan's Laws

Some equivalences (read $\equiv$ as "is equivalent to"):

1. DeMorgan: $\neg(A \wedge B) \equiv \neg A \vee \neg B$
2. Double negation: $\neg(\neg A) \equiv A$
3. DeMorgan: $\neg(A \vee B) \equiv \neg A \wedge \neg B$

A new proof method: we can use equivalences as *rewrite rules*. Proof of DeMorgan 3 from DeMorgan 1 and double negation:

$$
\begin{array}{ll}
\neg(\underline{A} \vee \underline{B}) & \\
= \neg(\underline{\neg(\neg A)} \vee \underline{\neg(\neg B)}) & \text{double double negation} \\
= \neg(\overline{\neg(\underline{\neg A})} \vee \overline{\neg(\underline{\neg B})}) & \\
= \neg(\underline{\neg(\neg A \wedge \neg B)}) & \text{DeMorgan 1} \\
= \neg(\overline{\neg(\underline{\neg A \wedge \neg B})}) & \\
= \neg A \wedge \neg B & \text{double negation}
\end{array}
$$

Satisfiability

Definition: A formula is *satisfiable* if at least one assignment evaluates to true.

A formula is satisfiable iff its negation is not valid. (DeMorgan's law in another form.)

Validity is kind of like "$\forall$".

Satisfiability is kind of like "$\exists$".

Determining whether a formula is satisfiable, efficiently, is a core problem in computer science. Examples: Solving puzzles, finding successful plans, arranging items in space, factoring, finding paths in graphs...

Checking satisfiability

Easy if few variables. Just write out the truth table!

P	Q	$\neg Q$	$\neg P$	$Q \vee \neg P$	$\neg Q \wedge (Q \vee \neg P)$
F	**F**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**
T	**F**	**T**	**F**	**F**	**F**
T	**T**	**F**	**F**	**T**	**F**

Blows up as the number of variables gets large. Need another way.

Disjunctive normal form

Definition: A formula in *disjunctive normal form* is an OR of terms, where each term is an AND of variables or negations of variables.

$(A \land B \land \neg C) \lor (\neg B \land C)$

$A \lor B \lor (A \land B \land \neg C)$

Not in DNF: $(A \land B) \lor \neg(B \land C)$

Disjunctive normal form is universal

Theorem: For every formula, there is an equivalent formula written in DNF.

Proof: You can read the terms off of the truth table, turning each “true” row into a conjunction of literals.

A	B	C	value		
F	**F**	**F**	**F**		
F	**F**	**T**	**T**	$\leftarrow$	$\neg A \wedge \neg B \wedge C$
F	**T**	**F**	**F**		
F	**T**	**T**	**F**		
T	**F**	**F**	**F**		
T	**F**	**T**	**T**	$\leftarrow$	$A \wedge \neg B \wedge C$
T	**T**	**F**	**T**	$\leftarrow$	$A \wedge B \wedge \neg C$
T	**T**	**T**	**F**		

$(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)$

Properties of disjunctive normal form

How big could the disjunctive normal form get? Big!

Definition: If every variable appears exactly once in every term in a disjunctive normal form expression, then it is in *full disjunctive normal form*.

Book	Wikipedia/me
disjunctive form	disjunctive normal form
disjunctive normal form	full disjunctive normal form

Given a formula in DNF (disjunctive normal form), can we determine whether it is satisfiable? Valid? Satisfiability is easy—a single term tells us a satisfying assignment. Validity is not obvious—a given term might exclude an assignment, but perhaps another picks it up? What if it's in full DNF? Then, we'd need a term for each "true" row in the truth table: valid iff 2^n rows.

Conjunctive normal form

Definition: A formula in *conjunctive normal form* is an AND of clauses, where each clause is an OR of variables or negations of variables.

$(\neg A \vee \neg B \vee C) \wedge (B \vee \neg C)$

$\neg A \wedge B \wedge (\neg A \vee C)$

Not an example:

Conjunctive normal form is universal

Theorem: For every formula, there is an equivalent formula written in CNF.

Proof: Negate the truth table. Write in DNF. Negate formula via DeMorgan's law. QED.

A	B	C	value	negated		
F	**F**	**F**	**T**	**F**		
F	**F**	**T**	**F**	**T**	$\leftarrow$	$\neg A \wedge \neg B \wedge C$
F	**T**	**F**	**T**	**F**		
F	**T**	**T**	**T**	**F**		
T	**F**	**F**	**T**	**F**		
T	**F**	**T**	**F**	**T**	$\leftarrow$	$A \wedge \neg B \wedge C$
T	**T**	**F**	**F**	**T**	$\leftarrow$	$A \wedge B \wedge \neg C$
T	**T**	**T**	**T**	**F**		

DNF for negated: $(\neg A \wedge \neg B \wedge C) \vee \ldots$

CNF: $(A \vee B \vee \neg C) \wedge \ldots$

Properties of conjunctive normal form

How big could the conjunctive normal form get? Big.

Definition: If every variable appears in every clause in a conjunctive normal form expression, then it is in *full conjunctive normal form*.

Book	Wikipedia/me
conjunctive form	conjunctive normal form
conjunctive normal form	full conjunctive normal form

Given a formula in CNF (conjunctive normal form), can we determine whether it is satisfiable? Valid? Validity is easy now—a single clause throws out an assignment, so a single clause makes the formula not valid. Satisfiability is not so clear—each clause knocks out some assignments, but not clear if the set of clauses miss anything.

Some algebraic rewrite rules

- Commutativity: $A \wedge B = B \wedge A, \quad A \vee B = B \vee A.$
- Associativity: $(A \wedge B) \wedge C = A \wedge (B \wedge C),$
  $(A \vee B) \vee C = A \vee (B \vee C).$
- Identity: $\mathbf{T} \wedge A = A, \quad \mathbf{F} \vee A = A.$
- Zero: $\mathbf{F} \wedge A = \mathbf{F}, \quad \mathbf{T} \vee A = \mathbf{T}.$
- Idempotence: $A \wedge A = A, A \vee A = A.$
- Complements: $A \wedge \neg A = \mathbf{F}, \quad A \vee \neg A = \mathbf{T}.$
- Distributivity:
  $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$

Converting to disjunctive normal form

$$\begin{aligned}
&\neg((A \wedge B) \vee (A \wedge C)) && \\
&= \neg(A \wedge B) \wedge \neg(A \wedge C) && \text{DeMorgan} \\
&= (\neg A \vee \neg B) \wedge (\neg A \vee \neg C) && \text{DeMorgan} \\
&= ((\neg A \vee \neg B) \wedge \neg A) \vee ((\neg A \vee \neg B) \wedge \neg C) && \text{Distributive} \\
&= (\neg A \wedge \neg A) && \\
&\quad \vee (\neg B \wedge \neg A) && \\
&\quad \vee (\neg A \wedge \neg C) && \\
&\quad \vee (\neg B \wedge \neg C) && \text{Distributive}
\end{aligned}$$

We converted the formula into disjunctive normal form. In fact, it's a general strategy for doing so: (1) use DeMorgan to push the "NOT"s all the way down to the propositions. (2) use distributivity to "multiply out" all the clauses. You'll get some practice at this in recitation.