Overview

1. Proof Rules
2. Introduction Rules
3. Elimination Rules
4. Negation Rules
Proof vs Truth

For today, we’re going to think about propositional formulas, like $p \land q \rightarrow r$.

We saw a way to evaluate when these kinds of formulas are true: truth tables.

Today, we’ll see a way to prove these kinds of formulas. Why the distinction?
Goals and Hypotheses

Suppose we have a proposition $G$ that we want to prove.

The structure of $G$ determines what we need to do to prove it.

Suppose we know a proposition $H$.

The structure of $H$ determines what we can do with this fact.

During a proof, we might have multiple things that we want to prove ($goals$). Associated to each goal, there is a list of things we know (a list of $hypotheses$, making up a $context$).
The Proof Game

Start: one goal, zero hypotheses.

Aim: all goals completed.

Moves: proof rules, to change proof state.

An example from last week: "there is a perfect square whose final digit is 4." Proof rule: to prove an existential, provide a witness: $8^2$. Goal becomes, "the final digit of $8^2$ is 4." (True by computation.)
Let’s make this more precise.

We introduced the language of propositional logic: formulas built out of atoms and connectives $\land$, $\lor$, $\neg$, $\to$, $\leftrightarrow$.

What are the proof rules associated with these symbols?
Two categories of rules. *Introduction rules* say how to prove a goal of a certain form. *Elimination rules* say how to use a hypothesis of a certain form.
"And" Introduction

If your goal is to prove $P \land Q$: first prove $P$, then prove $Q$. (Turns one goal into two smaller goals.)
"Or" Introduction

If your goal is to prove $P \lor Q$, there are two rules you can follow:

- Prove $P$. ("left")
- Prove $Q$. ("right")

Both rules turn one goal into one smaller goal.

An example: prove $(1 + 1 = 2 \lor 1 + 1 = 3) \land (2 \cdot 2 = 5 \lor 2 \cdot 2 = 4)$. 
Implication Introduction

To prove $P \rightarrow Q$: assume $P$ (a new hypothesis), and show $Q$ (a new goal).

Goal: if $x$ is even, then $x^2$ is even. Suppose $x$ is even. We use this fact to show that $x^2$ must be even.

To show $P \leftrightarrow Q$: show $P \rightarrow Q$ and $Q \rightarrow P$ (two goals).
Atoms

If you have a hypothesis $P$ in your context, you can close a goal of $P$. ("By assumption")

Goal: if $x$ is even, then $x$ is even. Suppose $x$ is even. Our goal is now to show that $x$ is even. This follows by assumption.
In Lean

Introduction rules in Lean:
- and intro: split_goal
- or intro: left, right
- implication intro: intro h
- iff intro: split_goal
"And" Elimination

If you know \( P \land Q \), you know two things:

- \( P \)
- \( Q \)

Yes, this sounds silly to say out loud. We usually don’t think about this.

In terms of proof state: turns one hypothesis into two smaller hypotheses.
"Or" Elimination

This one’s more interesting!

If you know $P \lor Q$, and your goal is $G$, you can *reason by cases*. That is: if you show $P \rightarrow G$, and you show $Q \rightarrow G$, then you have shown $G$.

In terms of proof state: creates two goals, each with a new hypothesis.
Implication Elimination: modus ponens

If \(x\) is prime, then \(x \geq 2\). \(x\) is prime. Therefore, \(x \geq 2\).

General pattern: if you know \(P \rightarrow Q\) and you know \(P\), then you know \(Q\).

Adds a hypothesis.

Alternate phrasing: if your goal is to show \(Q\), and you know \(P \rightarrow Q\), it suffices to show \(P\).

Changes the goal.

(Iff elimination is easy: if you know \(P \leftrightarrow Q\), then you know \(P \rightarrow Q\) and \(Q \rightarrow P\).)
Getting comfortable with contradiction

We live in a world where things make sense. (...)

In our sensible world, some statements are true and some are false. But none are true \textit{and} false.

So if we can prove that something is both true and false, we’re temporarily living in nonsense land. Anything follows.
Negation elimination and introduction

Elimination proof rule: if you know $P$ and you know $\neg P$, you can prove anything (i.e. close any goal).

Introduction proof rule: if your goal is to prove $\neg P$, you can assume $P$, and show "false". Proof by contradiction!
Example proof by contradiction

Proposition: $\sqrt{2}$ is not rational.

We prove that $\sqrt{2}$ is not rational by contradiction. Suppose $\sqrt{2}$ is rational. By the definition of “rational”, that means $\sqrt{2} = p/q$ where $p$ and $q$ are integers. Furthermore, we can choose $p$ and $q$ to be in lowest terms so they have no factors in common. Squaring both sides, we get $2 = p^2/q^2$ or $2q^2 = p^2$. Since $q^2$ is an integer, and $p^2$ is an integer times 2, $p^2$ is even. By a similar argument to the one for odd squares (from a few lectures ago), that means $p$ must be even. If $p$ is even, $p^2$ must be divisible by 4. Since $2q^2$ is divisible by 4, $q^2$ must be divisible by 2 (the other factor of two must be there). That means both $p$ and $q$ are even. But, then $p/q$ is not in lowest terms. Since we already asserted that $p/q$ is in lowest terms when $p$ and $q$ were chosen, we’ve reached a contradiction. Therefore, $\sqrt{2}$ must be irrational.