



Propositional Logic

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Overview

- 1 Propositions from Propositions (3.1)
 - NOT, AND, and OR (3.1.1)
 - IMPLIES (3.1.2)
 - If and Only If (3.1.3)

- 2 Translating into propositional logic



What is propositional logic?

We have:

- *Atoms*: propositions that we can't decompose any further
- *Connectives*: operators that take in propositions and produce new propositions

We build compound propositions out of smaller ones. "It is snowing, and I am cold."

"Boolean logic": every proposition is either true or false. The truth of a compound proposition depends on the truth of its components.



Notation for propositional logic

Our connectives:

- **NOT:** $\neg P$
- **AND:** $P \wedge Q$
- **OR:** $P \vee Q$
- **IMPLIES (IF-THEN):** $P \rightarrow Q$
- **IFF:** $P \leftrightarrow Q$

$$P \wedge \neg Q$$

$$P \wedge (\neg Q)$$

$$(A \wedge B) \rightarrow (C \vee D)$$



Truth tables

Truth tables help convey how to interpret logical statements.

P	$\neg P$
F	T
T	F

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Like a multiplication table, but flattened (so easier to apply to more inputs.)



OR

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

In English, often rendered as “ P or Q or both”. Inclusive or.

P	Q	$P \text{ XOR } Q$
F	F	F
F	T	T
T	F	T
T	T	F

Exclusive or, too.



IMPLIES (3.1.2)

Implication

Perhaps counterintuitive.

X	Y	$X \rightarrow Y$
F	F	T
F	T	T
T	F	F
T	T	T

This is known as the *material conditional*. Compare:

If it is raining outside, then I have my umbrella. (material)

If it were raining outside, then I would have my umbrella. (counterfactual)

Math deals with material conditionals, so this class does too.



Implication

X	Y	$X \rightarrow Y$
F	F	T
F	T	T
T	F	F
T	T	T

Claim: If p is a prime number greater than 2, then p is odd.

“Should” be true for every number p .

Definition: In “ $X \rightarrow Y$ ”, X is called the *hypothesis* and Y the *conclusion*.

Can you think of an equivalent way to write this using \wedge , \vee , \neg ?



Implication

X	Y	$X \rightarrow Y$
F	F	T
F	T	T
T	F	F
T	T	T

X	Y	$\neg X$	$\neg X \vee Y$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T



Truth table for iff

X	Y	$X \leftrightarrow Y$
F	F	T
F	T	F
T	F	F
T	T	T

Example: A number is divisible by three *if and only if* the sum of its digits is divisible by three.

Definitions have this form.



Truth tables for compound formulas

To build a truth table for a compound formula:

- Find the atoms, and make those columns.
- Find the compound subformulas, and make those columns.
- Fill in all combinations of truth values for the atoms.
- Propagate right, column by column.

An example in Lean



Natural language translations

Translating English sentences into propositional logic seems straightforward: choose your atomic propositions. Then "and" $\rightarrow \wedge$, "or" $\rightarrow \vee$, ...

But it's not always quite so easy. Define:

- $C :=$ "Rob likes cacti"
- $G :=$ "Rob likes geraniums"
- $F :=$ "Rob's garden is full"
- $W :=$ "Rob wants more plants"

"Rob likes cacti and geraniums." $C \wedge G$

"Rob wants more plants, but his garden is full." $W \wedge F$

"If Rob does not want more plants, then he doesn't like cacti or geraniums."

$\neg W \rightarrow \neg(C \vee G)$ or maybe $\neg W \rightarrow (\neg C \wedge \neg G)$