Homework 9

Due: Wednesday, April 26

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Alan and En-Hua each have *weighted* coins: when flipped, these coins land on heads with probability 1/3 and tails with probability 2/3. Suppose Alan and En-Hua each toss their coin twice. Let X be the number of heads Alan obtains and Y the number of heads En-Hua obtains. In all of the problems below, show the calculation that leads to your answer.

a. We want to model this situation as a probability space. Thinking in terms of sets: what is our *sample space* S? What are the *outcomes*? What is the *event* $X = 1$?

b. What is the distribution of X? (In other words: for each value v in the range of X, what is $Pr[X = v]$?) What is the distribution of Y?

c. What is the distribution of $X + Y$?

d. Given that $X + Y = 2$, find the distribution and expected value of X.

Solution:

a. The sample space is the set of sequences $(a_1, a_2, e_1, e_2) \in \{H, T\}^n$: that is, sequences of four coin flips. Each outcome is an element of this sample space. The event $X = 1$ is the subset of the sample space where exactly one of a_1 and a_2 is H. Since order doesn't matter in the questions we ask, we could also model it as pairs of sets $(\{a_1, a_2\}, \{e_1, e_2\})$.

b. X and Y have the same distribution:

- $P(X = 0) = 4/9$
- $P(X = 1) = 4/9$
- $P(X = 2) = 1/9$

c. The probabilities found in part a. can be used to calculate the joint probabilities of (X, Y).

The distribution of $X + Y$ is as follows:

- $P(X + Y = 0) = P(0, 0) = 16/81$
- $P(X + Y = 1) = P(1, 0) + P(0, 1) = 32/81$
- $P(X + Y = 2) = P(2, 0) + P(0, 2) + P(1, 1) = 24/81$
- $P(X + Y = 3) = P(2, 1) + P(1, 2) = 8/81$
- $P(X + Y = 4) = P(2, 2) = 1/81$

d. The distribution of X can be calculated from the joint probabilities and results in part b:

- $P(X = 0|X + Y = 2) = P(X = 0, Y = 2)/P(X + Y = 2) = \frac{4/81}{24/81} = 1/6$
- $P(X = 1|X + Y = 2) = P(X = 1, Y = 1)/P(X + Y = 2) = \frac{16/81}{24/81} = 2/3$
- $P(X = 2|X + Y = 2) = P(X = 2, Y = 0)/P(X + Y = 2) = \frac{4/81}{24/81} = 1/6$

$\mathbb{E}[X|X + Y = 2] = 0 \cdot 1/6 + 1 \cdot 2/3 + 2 \cdot 1/6 = 1.$

Rubric:

Problem 2

It is time for the annual Brown Horticultural Society's (BHS)[1] annual Great Garden Grow-off![2]

The format of the Grow-off is as follows: 64 teams compete head-to-head in a single elimination tournament (with the healthiest-looking plants declared as winners). Thus, 32 games will be played in the first round, 16 in the second round, etc. with winners moving on to subsequent rounds, for a total of 63 games. There are no ties.

You have been tasked with predicting a winner for each of the 63 games *before the competition starts*. You receive 1 point for each correctly guessed winner for the first round, with the points rewarded for each correct prediction doubling for each subsequent round—you receive 32 points for correctly guessing the winner of the final championship match in the 6th round.

There's a catch: you are an absolute gardening novice with no ability to determine the quality of the plants whatsoever, thus you've left each of your predictions to a fair coin flip, i.e. for a given match-up you pick a predicted winner with 50/50 odds. How many points do you expect to earn?

Hint: What is the probability of correctly guessing the winner of a game in round 1? round 2? What about in round n?

Solution:

The probability of guessing a correct winner in round 1 is 1/2, based on a fair coin flip. The probability of guessing a correct winner in round 2 is 1/4, the probability that we predict the winner will win round 1 and round 2—they must win round 1 to make it to round 2 ($1/2 * 1/2, P(Round2 \mid Round1)$. For any round n, the probability of correctly guessing the winner of a game is $1/2^n$ (given the prob of each win is 1/2, and thus the conditional probability just becomes a series of 1/2 * 1/2...).

A basic approach is to calculate the expected points earned for each round, as there are only 6 rounds to consider ($64 = 2^6$). The expected payoff for a round n may be calculated as follows:

$$\frac{64}{2^n} \cdot \frac{1}{2^n} \cdot 2^{n-1}$$

the number of games in a round multiplied with the probability of correctly guessing the correct winner of a game multiplied with the points awarded for a correct guess.

[1] Fictionalized organization.

[2] Fictionalized competition.

For rounds 1-6, the total number of expected points is

$$32 \cdot \frac{1}{2} \cdot 1 + 16 \cdot \frac{1}{4} \cdot 2 + 8 \cdot \frac{1}{8} \cdot 4 + 4 \cdot \frac{1}{16} \cdot 8 + 2 \cdot \frac{1}{32} \cdot 16 + 1 \cdot \frac{1}{64} \cdot 32 = 31.5$$

There is a more direct solution if we consider the expected payoff per game:

$$\frac{1}{2^n} \cdot 2^{n-1}$$

the probability of correctly guessing the correct winner of a game multiplied with the points awarded for a correct guess, where n is the round number. This expression simplifies to $1/2$. The expected payoff per game is constant regardless of round. Thus, the total number of expected points is

$$63 \cdot \frac{1}{2} = 31.5$$

the total number of games multiplied with the expected payoff per game.

Rubric:

Problem 3

You have a straight, 100cm length of vine, with thorns at each centimeter, from 0 to 100 inclusive. You dip it into liquid nitrogen and drop it. It breaks exactly halfway between two adjacent thorns, where the point of breakage is equally likely to be in between any pair of adjacent thorns. One piece must be longer than the other piece, since it couldn't have broken exactly at the middle thorn; call the shorter piece S and the longer piece L.

a. What's the expected number of thorns on S?

   To make this clearer, if we were talking about a length of vine with ten evenly spaced thorns on it and it broke between thorns 7 and 8, there would be 7 thorns on the longer piece, and the remaining three thorns would be on the shorter piece.

b. What about the expected number of thorns on L?

c. What is the expected value of the *product* of the number of thorns on S and the number of thorns on L? (Recall that we proved some formulas for sums, e.g. for $\sum_{i=0}^{n} i^2$, using induction earlier this semester. You do not need to re-prove these formulas.)

Solution:

a. Let k be the position of the break (in cm, rounded down). Since the vine is equally likely to break between any pair of adjacent thorns, the number of thorns on the shorter piece n is

$$n = \begin{cases} k+1, \text{if } 0 \leq k \leq 49 \\ 100-k, \text{otherwise} \end{cases}$$

   In the former case, n is equally likely to be 1 to 50. However, this is also true for the latter case! Then the expected number of thorns is $\frac{1+50}{2} = \frac{51}{2}$.

b. For the long end of the vine, notice that there are a total of 101 thorns on the ruler; since there are only 2 pieces, the long end must have all the thorns that are not on the short end. Then the expected number is $101 - \frac{51}{2} = \frac{151}{2}$.

c. Let i be the number of thorns on S.

   There are 101 thorns in total, so the product, $S(L)$ is $i(101-i)$.

We compute, using the formula for expected value:

$$\begin{aligned}
\sum_{i=1}^{50} i(101-i)/50 &= \frac{101}{50}\sum_{i=1}^{50} i - \frac{1}{50}\sum_{i=1}^{50} i^2 \\
&= \frac{101}{50}\frac{50\cdot 51}{2} - \left(\frac{1}{300}50\cdot 51\cdot 101\right) \\
&= 1717
\end{aligned}$$

Note in the above math we use the formula for a sum of i^2, and take out the $\frac{1}{6}$ in the intermediary step, but the answer we are looking for is **1717**.

Rubric:

a. 2 points for correct answer and justification

b. 2 points for correct answer and justification

c. 5 points for correct answer and justification

Problem 4 (Mind Bender — *Extra Credit*)

This week's mindbender is a Lean question!

The problem can be found by navigating to `BrownCs22/Homework/Hw9.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.