Homework 8

Due: April 19, 2023

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

a. For each of the following pairs of events, identify whether they are independent and justify why or why not.

   i. When flipping a fair coin three times:
      - the first coin is a tails
      - there is a run of exactly two heads (i.e. two, but not three, heads are flipped in a row)

   ii. When generating a 0/1 string of length 5:
      - the 3rd digit is a 1
      - there are at least two 0s

b. Suppose you have a bag of 4 balls. One ball is red, one ball is yellow, one ball is blue, and one ball is **red, yellow, AND blue**. Define X_1 as picking a red ball, define X_2 as picking a blue ball, and define X_3 as picking a yellow ball.

   i. Are the events X_1, X_2, and X_3 pairwise independent?

   ii. Are the events X_1, X_2, and X_3 triplewise independent?

Solution:

a. **Note:** The solution expects students to prove using $P(A) = P(A|B)$.
It is also acceptable for students to prove using $P(A \cap B) = P(A) \cdot P(B)$.

   i.

$$\begin{aligned} P(\text{run of two heads}) &= P(\text{HHT or THH}) \\ &= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \end{aligned}$$

$$= \frac{1}{4}$$

We can then look at the conditional probability.

$$\begin{aligned} P(\text{run of two heads}|\text{first flip tails}) &= P(\text{second flip heads} \cap \text{third flip heads}) \\ &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Since $P(\text{run of two heads})$ is equal to the conditional probability $P(\text{run of two heads}|\text{first coin is tails})$, the two events are independent.

ii. $P(\text{3rd digit is a 1}) = \frac{1}{2}$

$$\begin{aligned} P(\text{at least 2 zeros}) &= 1 - P(\text{exactly 0 zeroes}) - P(\text{exactly 1 zero}) \\ &= 1 - \binom{5}{0}\left(\frac{1}{2}\right)^5 - \binom{5}{1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 \\ &= 0.8125 \end{aligned}$$

We can then look at the conditional probability:

$$\begin{aligned} &P(\text{at least 2 zeroes} \mid \text{3rd digit is a 1}) \\ &= 1 - P(\text{exactly 0 zeroes in the other 4 digits}) - P(\text{exactly 1 zero in the other 4 digits}) \\ &= 1 - \binom{4}{0}\left(\frac{1}{2}\right)^4 - \binom{4}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 \\ &= 0.6875 \end{aligned}$$

Since $P(\text{at least 2 zeroes})$ is not equal to the conditional probability $P(\text{at least 2 zeroes}|\text{3rd digit is a 1})$, the two events are not independent.

b. i. These events are pairwise independent because $P(X_i) = P(X_i|X_k) = \frac{1}{2}\ \forall i, k \in \{1, 2, 3\}, i \neq k$.

ii. To examine triplewise independence, we note that $P(X_1 \cap X_2 \cap X_3)$ is the probability of picking a red, yellow, AND blue ball. There is only one way to do this, thus this evaluates to $\frac{1}{4}$.

However the probabilities of X_1, X_2, and X_3 are all $\frac{1}{2}$, so if we multiply $P(X_1) \cdot P(X_2) \cdot P(X_3)$ we get $\frac{1}{8}$.

$P(X_1 \cap X_2 \cap X_3) \neq P(X_1) \cdot P(X_2) \cdot P(X_3)$, therefore the events are not triplewise independent.

Rubric:

8 points total

a. i. 2 pt total: 1 for correct answer, 1 for justification

   ii. 2 pt total: 1 for correct answer, 1 for justification

b. i. 2 pt total: 1 for correct answer, 1 for justification

   ii. 2 pt total: 1 for correct answer, 1 for justification

Problem 2

a. Jen and Carmen are playing with a spinner numbered from 1 to 210, and experimenting with different rules. For this round, Jen chose to win when the spinner lands on a number that is not a multiple of 2, 3, or 7. What is the probability that Jen wins? (In other words, what is the probability that the number the spinner lands on does not have 2, 3, or 7 as prime factors?)

b. Jen and Carmen started choosing increasingly unfair rules for their spinner game, and got into a big fight. Their parents took away their spinner, but now they have found ten blocks with the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ on each of them. In true child fashion, they have found a way around the rules. Now, they will draw the blocks randomly from a bag and line them up in the order they were chosen. Carmen wins if their sequence contains the sequence 024 or the sequence 456. (Note that this is an inclusive "or.") What is the probability Carmen wins?

   P.S. This is all theoretical. In reality, since they are sitting on opposite sides of the sequence of blocks, they will perceive the sequences as the reverse of what the other perceives, and another fight is inevitable.

Solution:

a. Start with 210 total numbers. Subtract the numbers divisible by 2, by 3, and by 7 (ignoring overcounting) to get $210 - \frac{210}{2} - \frac{210}{3} - \frac{210}{7} = 210 - 105 - 70 - 30 = 5$.

   Then add back in numbers subtracted twice- i.e. those divisible by both 2 and 3, both 3 and 7, or both 2 and 7 to get $5 + \frac{210}{6} + \frac{210}{21} + \frac{210}{14} = 5 + 35 + 10 + 15 = 65$.

   However, we have again overcounted the numbers divisible by all three, so we must subtract them again to get $65 - \frac{210}{42} = 65 - 5 = 60$.

   Therefore the total number not divisible by neither 2, 3, nor 7 is 60. The probability Jen wins of $\frac{60}{210} = \frac{2}{7}$

b. **Answer**: $\frac{2*8!-6!}{10!} = \frac{37}{1680}$.

   Justification: By the inclusion-exclusion principle, the number of permutations of the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ that contain the sequence 024 or the sequence 456 equals (number of permutations that contain the sequence 024) + (number of permutations containing the sequence 456) - (number of permutations containing the sequence 024 and the sequence 456).

   The number of permutations that contain the sequence 024 can be counted by fixing 024 into an indivisible group and counting the number of permutations that contain 024 and the 7 other digits. There are 7! different ways to reorder the 7 other digits. For each reordering of the 7 other digits, we can insert the

024 group of digits anywhere into that length-7 string. A string of length 7 has 8 different locations in which the 024 group can be inserted. So, the number of permutations that contain the sequence 024 is $8 * 7! = 8!$.

An identical argument with the group being 456 rather than 024 shows that the number of permutations that contain the sequence 456 is also $8 * 7! = 8!$.

The number of permutations containing the sequence 024 and the sequence 456 can be counted in a similar manner. However, note that a permutation contains both the sequence 024 and the sequence 456 iff it contains the sequence 02456. Then, if we fix 02456 into an indivisible group, there are 5 remaining digits that can be reordered in 5! different ways, and 6 different locations in which the group can be inserted into the string of those 5 remaining digits. So, the number of permutations containing the sequence 024 and the sequence 456 is $6 * 5! = 6!$.

Therefore, the number of permutations of the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ that contain the sequence 024 or the sequence 456 equals $8! + 8! - 6! = 2 * 8! - 6! = 79920$.

The total number of sequences is $10! = 3628800$.

Thus, the probability that Carmen wins is $\frac{37}{1680}$.

Rubric:

TODO

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Hw8.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.

Rubric:

Question 3 Autograded on Lean/Gradescope.

Problem 4

This is a plant-themed version of a popular puzzle that made its rounds through math and computer science circles back in 2001:

Imagine a game with three players. Each player has either a rose or a tulip taped to their head, determined independently and uniformly at random, i.e. each player has 50/50 odds of receiving a rose or a tulip. Each player can see the flowers on the other two players heads, but cannot see their own.

After observing each other's flowers, players *simultaneously* guess their flower type or pass (choose not to guess). There is no communication of any form during the game, but the players may devise a strategy ahead of time.

The players collectively win the game if at least one player guesses the type of flower on their head correctly, and no players guess incorrectly. For instance, if one player guesses correctly and the others pass, then everyone wins. On the other hand, if two players guess correctly, but the last players guesses incorrectly, then everyone loses.

One strategy, for example, would be to have player one always guess tulip and have the other two players always pass. This strategy wins 50% of the time.

Devise a strategy that maximizes the players' odds of winning this game. What are their odds using this strategy? Show how you computed this value.

Solution:

The optimal strategy is as follows, from this NYT article:

"Once the game starts, each player looks at the other two players' [flowers]. If the two [flowers] are different [types], he passes. If they are the same [type], the player guesses his own [flower] is the opposite [type]." Note the problem originally uses red and blue hats.

There are 8 possible permutations of flower types among the three people, all equally likely. It is always the case that at least two people share the same flower type (Pigeonhole Principle). In 6 cases, two people share the same flower type and the third person has a differing flower type. In only 2 cases, all three people have the same flower type. Thus, we see that this strategy wins 3/4 of the time.