Homework 5

Due: March 9, 2023

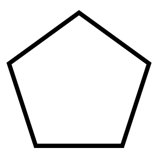
All homeworks are due at 11:59 PM on Gradescope.

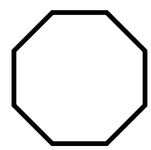
Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

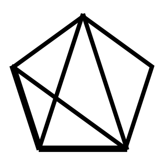
Problem 1

A *regular n-gon* is a polygon with n sides, all of equal length, and n angles, all of equal measure. For example, a square is a regular 4-gon, and the images below are a regular 5-gon and 8-gon:





A *diagonal* of an n-gon is a line connecting two non-adjacent vertices. For instance, here are three diagonals of the regular 5-gon:



Show using induction that for all $n \in \mathbb{N}$ where $n \geq 3$, a regular n-gon always has $\frac{(n)(n-3)}{2}$ diagonals.

Solution:

The proof is by weak induction on n with starting point $n = 3$.

Base Case: Take $n = 3$, so we're working with a triangle. Since a triangle has no diagonals, we expect the formula to yield 0. And it does, since $\frac{3(3-3)}{2} = 0$.

Inductive Step: We assume as our induction hypothesis that $P(n)$ holds, i.e., that an n-gon has $\frac{(n)(n-3)}{2}$ diagonals.

Now, observe that we can create an $(n+1)$-gon by taking an n-gon and adding a side (which leads to a new vertex). This adds $n-2$ diagonals (one from the new vertex to each of the vertices to which it is not adjacent), and also turns one of the sides from the n-gon into a diagonal (the side between the vertices between which we inserted our new vertex). Thus, the new shape has $\frac{(n)(n-3)}{2} + n - 2 + 1$ diagonals. And then notice that

$$\begin{aligned}
\frac{(n)(n-3)}{2} + n - 2 + 1 &= \frac{(n)(n-3)}{2} + n - 1 \\
&= \frac{n^2 - 3n}{2} + n - 1 \\
&= \frac{n^2 - n - 2}{2} \\
&= \frac{(n+1)(n-2)}{2}
\end{aligned}$$

so that $P(n+1)$ holds.

The result follows by induction.

Problem 2

The TAs are purchasing some plants to decorate the front of the DeCiccio Auditorium stage during CS 22 lectures. The front of the stage can hold up to 100 pounds; the TAs, being floral aficionados, would like to make maximal use of this carrying capacity.

The CS 22 Plant Emporium sells two types of potted plants: bulbs, which each weigh 8 pounds; and succulents, which each weigh 18 pounds.

a. Determine how many of each plant the TAs should purchase if they would like to maximize the total *weight* of the plants they buy. (Remember that they cannot exceed the carrying capacity of the stage.) Justify your answer.

b. But wait, the CS 22 Plant Emporium has an innovative new product that defies the laws of physics: the *anti-plant*(™)! For every plant species the Emporium sells, they now also sell an *anti-species* that has a weight equal to the *negative* of that of the original plant (e.g., an anti-bulb weighs -8 pounds, and an anti-succulent weighs -18 pounds).

   Now that the TAs can purchase anti-plants as well as traditional ones, describe all possible configurations of plants (including anti-plants) they can buy if they want their purchase to weigh exactly 100 pounds. Prove that these configurations all weigh exactly 100 pounds, and that they are the only possible solutions.

   Example

   If succulents weighed 97 pounds and bulbs weighed 3 pounds, we could describe all possible configurations as follows: for any $x \in \mathbb{Z}$, we purchase $1+3x$ succulents (or anti-succulents if this number is negative) and $1-97x$ bulbs (or anti-bulbs).

You may cite without proof the following lemma: for all $a, b, c \in \mathbb{Z}$, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$

Solution:

a. $8 \cdot 8 + 2 \cdot 18 = 100$, so 8 bulbs and 2 succulents.

b. Solutions are of the form $8 - 9x$ bulbs (negative = anti-bulbs) and $2 + 4x$ succulents (negative = anti-succulents) for $x \in \mathbb{Z}$.

Problem 3

The Fibonacci numbers are defined recursively, according to the following rules:

$$f_0 = 1$$
$$f_1 = 1$$
$$f_{n+2} = f_n + f_{n+1}$$

Example

$f_2 = f_0 + f_1 = 2$. $f_3 = f_1 + f_2 = 3$. $f_4 = f_2 + f_3 = 5$.

Prove using induction that for every n, f_n and f_{n+1} are relatively prime numbers.

You will likely find it useful to prove that for all $a, b \in \mathbb{Z}$, $\gcd(a, b + a) = \gcd(a, b)$.

Solution:

We first require the following lemma: for all $a, b \in \mathbb{Z}$, we have $\gcd(a, b{+}a) = \gcd(a, b)$.

Proof of Lemma. Let $d = \gcd(a, b)$. So $d \mid a$ and $d \mid b$, so there exist $j, k \in \mathbb{Z}$ such that $a = dj$ and $b = dk$. Then $a + b = dj + dk = d(j + k)$, so $d \mid a + b$ (and we already know $d \mid a$). So d is a common divisor of a and $a + b$.

It now remains to show that d is the *greatest* common divisor of a and $a + b$. To do so, we take an arbitrary common divisor d' of a and $a + b$ and show that it must be no greater than d.

So let $d' \in \mathbb{Z}$ be such that $d' \mid a$ and $d' \mid a + b$. We can write $a = rd'$ and $a + b = sd'$ for some $r, s \in \mathbb{Z}$. Then since $(a + b) - a = b$, we have that $b = sd' - rd' = d'(s - r)$, so that $d' \mid b$. Now we have that $d' \mid a$ and $d' \mid b$, and since we already know that $d = \gcd(a, b)$, we must have that $d' \leq d$ by the definition of gcd. And this means that d is the gcd of a and $b + a$ as well. $\square$

We now turn to the main proof, which is by induction on n.

Base Case: Take $n = 0$. We have $f_0 = 1$ and $f_1 = 1$. Notice $\gcd(1, 1) = 1$, so that f_0 and f_1 are relatively prime.

Inductive Step: Fix some $n \in \mathbb{N}$ and assume as our induction hypothesis that f_n and f_{n+1} are relatively prime.

Now observe that

$$\begin{aligned}
\gcd(f_{n+1}, f_{n+2}) &= \gcd(f_{n+1}, f_{n+1} + f_n) && \text{(Definition of } f_{n+2}\text{)} \\
&= \gcd(f_{n+1}, f_n) && \text{(Since } \gcd(a, b + a) = \gcd(a, b)\text{)} \\
&= 1 && \text{(Induction Hypothesis)}
\end{aligned}$$

It thus follows that f_{n+1} and f_{n+2} are relatively prime, so the claim holds for $n + 1$.

The result follows by induction.

Problem 4 (Mind Bender — *Extra Credit*)

Let $\langle a_k \rangle_{k \in \mathbb{N}}$ be the sequence of natural numbers defined as follows:

- $a_0 = 0$.
- $a_1 = 1$.
- For all natural $k \geq 2$, $a_k = 2a_{k-1} + a_{k-2}$.

a. Show that for all $n \in \mathbb{N}$ such that $n \geq 1$, we have $\sum_{k=0}^{n} a_k < 2a_n$.

b. Show that we can write any number $n \in \mathbb{N}$ as the sum/difference of elements of the sequence, i.e., $n = a_{j_1} \pm a_{j_2} \pm \cdots \pm a_{j_r}$ for some indices $j_1, j_2, \ldots, j_r$.

(More formally, we are asking you to prove that for any $n \in \mathbb{N}$, there exist some

- number of terms $r \in \mathbb{N}$,
- indices $\{j_s \mid s \in \mathbb{N} \text{ and } s < r \text{ and } \forall s \in \mathbb{N}, j_s \in \mathbb{N}\}$, and
- exponents $\{p_s \in \{0,1\} \mid s \in \mathbb{N} \text{ and } s < r\}$

for which $n = \sum_{s=0}^{r-1} (-1)^{p_s} a_{j_s}$.)

You may cite without proof the fact that the sequence is strictly increasing for $k \geq 1$. You may also make use of your result in the preceding part.

Before trying to prove this, pick some particular values of n and try to write them as sums/differences of the terms of the sequence, and look for patterns as you do so. (You might find it useful to pick consecutive values of n.) In particular, consider whether each natural number n is less than, greater than, or equal to the sum of all sequence elements strictly less than it. How does this relate to whether you need to add terms or subtract them to obtain n?

Solution:

a. We prove $P(n) := \sum_{k=0}^{n} a_k < 2a_n$ for all $n \in \mathbb{N}^+$ by weak induction on n.

Base Case: Take $n = 1$. Then

$$\sum_{k=0}^{n} a_k = a_0 + a_1 = 0 + 1 = 1 < 2 = 2n,$$

so $P(1)$ holds.

Inductive Step: Fix some $n \in \mathbb{N}^+$. Our induction hypothesis is that $P(n)$ holds.

We then have

$$\begin{aligned} \sum_{i=0}^{n+1} a_i &= \sum_{i=0}^{n} a_i + a_{n+1} && \text{(Definition of summation)} \\ &< 2a_n + a_{n+1} && \text{(Induction Hypothesis)} \\ &\leq 2a_n + a_{n+1} + a_{n-1} && \text{(Our sequence is non-negative)} \\ &= a_{n+1} + (2a_n + a_{n-1}) \\ &= a_{n+1} + a_{n+1} \\ &= 2a_{n+1} \end{aligned}$$

so that $P(n+1)$ follows.

b. We first provide some intuition for this problem. Write out the first few terms of the sequence: 0, 1, 2, 5, 12, 29, 70.... Now consider the first few sums; we also list the next-lowest and next-highest sequence elements (for non-elements of the sequence), as well as the sum of all lower sequence elements:

- 0: N/A, $0 = 0$
- 1: N/A, $1 = 1$
- 2: N/A, $2 = 2$
- 3: 2, 5, 3, $3 = 2 + 1$
- 4: 2, 5, 3, $4 = 5 - 1$
- 5: N/A, $5 = 5$
- 6: 5, 12, 8, $6 = 5 + 1$
- 7: 5, 12, 8, $7 = 5 + 2$
- 8: 5, 12, 8, $8 = 5 + 2 + 1$
- 9: 5, 12, 8, $9 = 12 - 2 - 1$
- 10: 5, 12, 8, $10 = 12 - 2$
- 11: 5, 12, 8, $11 = 12 - 1$
- 12: N/A, $12 = 12$

Notice the emerging pattern (letting a_k be such that $a_k < n < a_{k+1}$): when we're less than the accumulated sum, we can always add to a_k some number that we've already managed to generate using terms no later than a_{k-1}; but as soon as we're greater, we start subtracting terms we've already generated from a_{k+1}. This motivates our proof strategy below.

The proof is by strong induction.

Our inductive claim needs to be strengthened. In addition to the problem statement, we require that for values n not in the sequence, the maximal-index term in its summation is a_{k+1} when $n > \sum_{i=0}^{k} a_i$ and a_k otherwise, where a_k is the sequence element such that $a_k < n < a_{k+1}$.

Base Cases: For $n = 0$, we can write $0 = a_0$. For $n = 1$, we can write $1 = a_1$. Since both of these are terms in the sequence, this is all we need to show.

Inductive Step: For the inductive step, we take $n \in \mathbb{N} \setminus \{0, 1\}$ and suppose that the claim holds for all $k < n$. Now consider two cases.

If n is an element of the sequence, then the sum is simply that element and we're done.

If n is not an element of the sequence, then we take a_k such that $a_k < n < a_{k+1}$. We know such a_k exists because the a_i are strictly increasing from a_1 onward and $a_1 = 1 < n$.

Now consider three cases.

If $n = \sum_{i=0}^{k} a_i$, then we've found a summation for n and we're done.

If $n > \sum_{i=0}^{k} a_i$, then observe that by part (a), we know that $n > 2a_k$, and thus $a_{k+1} - n < a_{k+1} - 2a_k < a_{k-1}$, and clearly $a_{k-1} < n$ since a_{k-1} is a term in a sum of nonnegative values that yields n. So the inductive hypothesis applies to $a_{k+1} - n$ and gives us a sum S. Note that since $a_{k+1} - n < a_{k-1}$, we know a_{k+1} doesn't appear in S by the IH, so we can produce the summation $a_{k+1} - S$ (i.e., just flip all the powers in S and add a_{k+1}). It just remains to check that a_{k+1} is the maximal-index term in this summation, and indeed it is (since the maximal-index term in S is a_k).

Lastly, if $n < \sum_{i=0}^{k} a_i$, then take $S = n - a_k$ to be the summation guaranteed by the IH (since $n > 1$ and the sequence is strictly increasing, we know that $a_k \geq 1 > 0$ and so $n - a_k < n$). Observe that since $n - a_k < \sum_{i=0}^{k-1} a_i$, its maximal-index term is at most a_{k-1}, so in particular a_k does not appear in S and so we can produce the sum $S + a_k = n$. Lastly, we observe that the maximal-index term in this summation is a_k (the maximal-index term in S is a_{k-1}), as required.

The result follows by strong induction.