Homework 3

Due: Feb 22, 2023

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

LaTeX tips

Here are some more LaTeX commands that might help you

`\subseteq` $\subseteq$ `\sqrt{a}` $\sqrt{a}$ `\simeq` $\simeq$
`\{a\mid a\in\Z}` $\{a \mid a \in \mathbb{Z}\}$

LaTeX *is* required starting with this homework. You should be using the `cs22` document class. Look for a LaTeX guide here.

Problem 1

For each proposition below, determine whether it is true for arbitrary sets A and B. If it is true, prove it! If it is false, provide a counterexample: that is, give two sets A and B for which the claim is false, and explain why it is false.

a. $|A \cup B| = |A| + |B|$

b. $|A \setminus B| = |A| - |B|$

c. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

Solution:

a. *Claim.* $|A \cup B| = |A| + |B|$

Proof. To disprove the above claim, the following is a counterexample to the claim. Consider $A = \{1, 2, 3\}$ and $B = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$. Hence, $|A \cup B| = 3$. Based on the claim, $|A \cup B| = |A| + |B| = 3 + 2 = 5$. However, $3 \neq 5$. Therefore, the given claim is false. $\square$

b. *Claim.* $|A \setminus B| = |A| - |B|$

Proof. To disprove the above claim, the following is a counterexample to the claim. Consider $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$. Then $A \setminus B = \{1, 2, 3\}$. Hence, $|A \setminus B| = 3$. Based on the claim, $|A \setminus B| = |A| - |B| = 4 - 3 = 1$. However, $3 \neq 1$. Therefore, the given claim is false. $\square$

c. *Claim.* $\mathcal{P}(A \cap B) = |Pow(A) \cap \mathcal{P}(B)$

Proof. To prove that the above claim is true, we will use set element method, i.e., we will prove that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$, which implies that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Sub-Claim 1: $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

Proof. Consider a set x where $x \in \mathcal{P}(A \cap B)$. This means that x is a subset of $A \cap B$. Furthermore, it implies that the x is a subset of A and a subset of B. Hence, $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$. This implies that $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Therefore, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. $\square$

Sub-Claim 2: $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

Proof. Consider a set x where $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. It means that x is a subset of A and a subset of B. This implies that x is a subset of $A \cap B$, which is synonymous to the following statement: $x \in \mathcal{P}\,(A \cap B)$. Therefore, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$. $\square$

By proving the above sub-claims, based on the definition of set equality, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. $\square$

Problem 2

When there is a *bijection* between two finite sets, we can conclude that those sets have the same number of elements. (The mindbender below explores what happens with infinite sets.)

Let A be a set with n elements. Let T be the set of all ordered pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Define a bijection between T and S (and prove that it is a bijection).

Conclude that T and S must be the same size.

Solution:

We will construct a bijection between T and S to show that they are of the same size. We begin by defining our mapping, and we then show that our mapping is in fact bijective.

To define our mapping, we'll first need to lay out some notation.

Let the n elements in A be denoted as $a_1, a_2, ...a_n$. Let the ith character in a string s in S be denoted as s_i. Note that we index from 1; that is, the first character in s is s_1.

We are now ready to define our mapping, f, from T to S.

We define our function as $f((X, Y)) = s_1 s_2 \dots s_n$, where

$$\begin{cases} s_i = 0 & \text{if } a_i \notin X \text{ and } a_i \notin Y \\ s_i = 1 & \text{if } a_i \notin X \text{ and } a_i \in Y \\ s_i = 2 & \text{if } a_i \in X \text{ and } a_i \notin Y \\ s_i = 3 & \text{if } a_i \in X \text{ and } a_i \in Y \end{cases}$$

For any input (X, Y), we generate an output that is a string of length n, where each character is a 0, 1, 2, or 3. We therefore can conclude that for any input (X, Y), $f((X, Y))$ is in S (that is, our mapping is well-defined).

We will now show that our mapping is bijective. To do so, we will first show that it injective, and we will then show that is surjective.

Injectivity:

We want to show that f is injective. To do so, our game plan is as follows:

i. We will consider an arbitrary (X_1, Y_1) and (X_2, Y_2) such that $f((X_1, Y_1)) =$

$f((X_2, Y_2))$. We will not make any assumptions about the equality of (X_1, Y_1) and (X_2, Y_2).

ii. However, we will then show that because $f((X_1, Y_1)) = f((X_2, Y_2))$, (X_1, Y_1) is forced to be equal to (X_2, Y_2).

Consider (X_1, Y_1) and (X_2, Y_2) such that $f((X_1, Y_1)) = f((X_2, Y_2))$. Call the string $f((X_1, Y_1))$ $s1$ and the string $f((X_2, Y_2))$ $s2$. Because $s1 = s2$, we know that each character of $s1$ must equal each character of $s2$; that is, $s1_i = s2_i$ for all i.

We know each $s1_i$ (and each $s2_i$) can be either 0, 1, 2, or 3.

If $s1_i = 0$ and hence $s2_i = 0$, then we know that: 1) a_i is not in X_1 and it is also not in X_2, and 2) a_i is not in Y_1 and it is also not in Y_2.

If $s1_i = 1$ and hence $s2_i = 1$, then we know that: 1) a_i is not in X_1 and it is also not in X_2, and 2) a_i is in Y_1 and it is also in Y_2.

If $s1_i = 2$ and hence $s2_i = 2$, then we know that: 1) a_i is in X_1 and it is also in X_2, and 2) a_i is not in Y_1 and it is also not in Y_2.

If $s1_i = 3$ and hence $s2_i = 3$, then we know that: 1) a_i is in X_1 and it is also in X_2, and 2) a_i is in Y_1 and it is also in Y_2.

But then we have that for each a_i, a_i is either in both X_1 and X_2, or in neither X_1 nor X_2. Hence, $X_1 = X_2$. Similarly, we have that for each a_i, a_i is either in both Y_1 and Y_2, or in neither Y_1 nor Y_2. Hence, $Y_1 = Y_2$.

But if $X_1 = X_2$ and $Y_1 = Y_2$, then we know that (X_1, Y_1) and (X_2, Y_2) must be equal.

Hence, we have that $f((X_1, Y_1)) = f((X_2, Y_2))$ implies that $(X_1, Y_1) = (X_2, Y_2)$, so f is injective.

Surjectivity:

We want to show that given any f is surjective. To do so, our game plan is as follows:

i. Given an arbitrary s in S, we will construct an (X, Y) dependent on this s, and we will justify that it is in our domain T.

ii. We will then justify that this (X, Y), when plugged into f, in fact leaves us with our original s. That is, we will justify that $f((X, Y)) = s$.

Consider an arbitrary s in S. We will construct a set X and a set Y as follows. For each i,

if $s_i = 0$, do not put a_i in X, and do not put a_i in Y.

if $s_i = 1$, do not put a_i in X, but put a_i in Y.

if $s_i = 2$, put a_i in X, but do not put a_i in Y.

if $s_i = 3$, put a_i in X, and put a_i in Y.

Now, consider (X, Y). From our description above, X contains 0 or more elements from A, and the same goes for Y. Hence, X and Y are subsets of A, and so we know that (X, Y) must be in T.

Consider $f((X, Y))$, and call it s'. By our mapping, we know that for any i,

$s'_i = 0$ when $a_i \notin X$ and $a_i \notin Y$. But by our description above, we know that if $a_i \notin X$ and $a_i \notin Y$, $s_i = 0$. Hence, if $s'_i = 0$, $s_i = 0$.

$s'_i = 1$ when $a_i \notin X$ and $a_i \in Y$. But by our description above, we know that if $a_i \notin X$ and $a_i \in Y$, $s_i = 1$. Hence, if $s'_i = 1$, $s_i = 1$.

$s'_i = 2$ when $a_i \in X$ and $a_i \notin Y$. But by our description above, we know that if $a_i \in X$ and $a_i \notin Y$, $s_i = 2$. Hence, if $s'_i = 2$, $s_i = 2$.

$s'_i = 3$ when $a_i \in X$ and $a_i \in Y$. But by our description above, we know that if $a_i \in X$ and $a_i \in Y$, $s_i = 3$. Hence, if $s'_i = 3$, $s_i = 3$.

But then we have that for any i, $s'_i = s_i$, which means that $s' = s$.

Hence, we have that $f((X, Y)) = s$, and therefore f is surjective, as needed.

We have now shown that f is injective and surjective, and hence we have that f is bijective. We therefore can conclude that T and S are of the same size!

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Hw3.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, copy the entire file to your computer, and upload it to Gradescope.

Question 3 Autograded on Lean/Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Mind Benders are extra credit problems intended to be more challenging than usual homework problems and are an exploration into a topic not covered in lecture. This week, we have an exploration into a concept called *equinumerosity* and *countability*.

Thought you learned how to count years ago? Think again!

This question explores how we can use the tools we've learnt, such as functions, to define a more general notion of counting. Especially in the case of counting *infinitely* many things.

Let A and B be sets, not necessarily finite. We say that A and B are *equinumerous* if there exists a bijective function $f : A \to B$. We use the notation $A \simeq B$ to mean "A is equinumerous to B."

Example

$\{a, b, c, d\}$ is equinumerous to $\{0, 1, 2, 3\}$ (what function f can we find?). $\{a, b, c, d\}$ has four elements.

In general, A has n elements if $A \simeq \{0, \dots, n-1\}$. We say that a set is *finite* if it has n elements, for some $n \in \mathbb{N}$.

But what about infinite sets? We say that A is *countably infinite* if $\mathbb{N} \simeq A$ (that is, we can find a bijective function $f : \mathbb{N} \to A$)[1]. But equinumerousity of infinite sets can be confusing!

a. Show that $\mathbb{N}^+$ (the set $\{1, 2, \dots\}$) is countably infinite: that is, show that there is a bijective function $f : \mathbb{N} \to \mathbb{N}^+$.

   This is not immediately obvious, since $\mathbb{N}^+ \subset \mathbb{N}$! That is, $\mathbb{N}$ contains 1 more element than $\mathbb{N}^+$ but they are still equinumerous!

b. Is it true that $\mathbb{Z}$ is countably infinite? Why or why not? If it is, prove so (that is, find a bijection $f : \mathbb{N} \to \mathbb{Z}$). If it isn't, prove such a bijection doesn't exist.

[1] Intuitively, this makes a bit of sense. We think of $\mathbb{N}$ to be the counting numbers, so if we can attach a 'count' to each element in A, even if A is infinite, it is 'countable' in some sense

c. Show that $\mathbb{N} \times \mathbb{N}$, the set of pairs of natural numbers, is countably infinite.

> **Hint:** Writing down an explicit bijection f is hard. You can draw a picture here, or explain precisely how you would "count" all of the pairs.

d. We give you these two facts, that you do not need to prove yourself:

   i. The relation $\simeq$ is transitive: if $A \simeq B$ and $B \simeq C$, then $A \simeq C$.
   ii. If $A \subseteq B$, A is not finite, and B is countably infinite, then A is countably infinite.

   Using the above two facts and part c., show that the set of positive rational numbers $\mathbb{Q}^+$ is countably infinite. Can you also conclude that $\mathbb{Q}$ is also countably infinite?

e. It's starting to sound like a lot of sets are countably infinite! Can you think of an infinite set that is *not* countable? You don't need to prove that it it is uncountable, but indicate why you think it isn't.

The following aren't for credit, but are some extra food for thought:

f. What set did you come up with? Can you prove that it is somehow not countably infinite?

g. You might wonder whether *all* uncountably infinite sets are equinumerous? Is there anything *even bigger* than uncountable infinity? Can you think of two uncountably infinite sets that aren't equinumerous. How would you prove so?

> You might want to look up *diagonalization* and *Cantor's Theorem*.

Solution:

(a) We can define a bijective function $f : \mathbb{N} \to \mathbb{N}+$ for all $n \in \mathbb{N}$ as:

$$f(n) = n + 1$$

To prove bijectivity, we must prove that the function is both injective and surjective.

We first show injectivity: for all $a, b \in \mathbb{N}$, if $f(a) = f(b)$, then $a = b$. Let $a, b \in \mathbb{N}$ such that $f(a) = f(b)$. So $a + 1 = b + 1$. Then it immediately follows that $a = b$ by subtracting 1 from both sides.

We next show surjectivity: for all $b \in \mathbb{N}^+$, there is some $a \in \mathbb{N}$ for which $f(a) = b$. Let $b \in \mathbb{N}^+$. Observe that since $b \neq 0$, there exists some number $n \in \mathbb{N}$ such that $b = n + 1$ by the definition of the naturals. Thus, $f(n) = b$ by the definition of f, and so we have found an element of $\mathbb{N}$ that maps to b under f. Surjectivity follows.

Because the function is both injective and surjective, the function must be bijective, and so equinumerosity follows.

(b)

$$f : \mathbb{N} \to \mathbb{Z}$$
$$f(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil$$

Note: other possible bijections include

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{-(n+1)}{2} & n \text{ odd} \end{cases}$$

and

$$f(n) = \begin{cases} \frac{-n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases}$$

We show that f is bijective by showing that it is injective and surjective.

We first show that f is injective.

Let $f(a) = f(b)$. Then $(-1)^a \lceil \frac{n}{2} \rceil = (-1)^b \lceil \frac{n}{2} \rceil$. If this value is negative, then a and b are odd so $\frac{a+1}{2} = \frac{b+1}{2} \to a = b$. If this value is positive, then a and b are even so $\frac{a}{2} = \frac{b}{2} \to a = b$.

We next show that f is surjective.

Choose some $z \in \mathbb{Z}$. If z is positive, let $n = 2z$, so $f(n) = (-1)^{2z} \lceil \frac{2z}{2} \rceil = z$. If z is negative, let $n = -2z - 1$, so $f(n) = (-1)^{-2z-1} \lceil \frac{-2z-1}{2} \rceil = (-1)\frac{-2z}{2} = z$.

The desired equinumerosity follows from the existence of the bijection f.

(c)

(The upper table is the table on which we are performing our enumeration; the lower one is the order in which we are enumerating the corresponding elements of the upper table. Elements numbered 5, 6, 9, 10, and 11 and all higher elements are not shown, hence their omission in the enumeration order.)

We enumerate pairs by proceeding down diagonals, starting in the upper-left corner and proceeding downward and to the left (the initial elements are $(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),\ldots)$. Since every diagonal has finite length, we will enumerate any element of $\mathbb{N}\times\mathbb{N}$ in finite time (i.e., the mapping is surjective); moreover, since all elements of the table are unique and are visited only once, the mapping is injective.

(d) Let $B = \left\{(a,b)\in\mathbb{N}\times\mathbb{N} \mid \frac{a}{b} \text{ is in reduced form}\right\}$. We show that $\mathbb{Q}^+ \simeq B$ by showing that the mapping $f : B \to \mathbb{Q}^+$ defined via $f(a,b) = \frac{a}{b}$ is bijective.

Surjectivity: Let $q \in \mathbb{Q}^+$. We know that q can be expressed as the reduced ratio of natural numbers $\frac{a}{b}$. But then since $q = \frac{a}{b}$ and $(a,b)\in\mathbb{N}\times\mathbb{N}$, we immediately see that $f(a,b) = \frac{a}{b} = q$.

Injectivity: Let $(a_1,b_1),(a_2,b_2)\in B$ such that $f(a_1,b_1) = f(a_2,b_2)$. Then $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. Note that both fractions are in reduced terms by the definition of B. But we know that fractions have unique lowest-terms representations, so $a_1 = a_2$ and $b_1 = b_2$, so that $(a_1,b_1) = (a_2,b_2)$, as desired.

By part (c), we know that $\mathbb{N}\times\mathbb{N}$ is countably infinite. Moreover, we see that $B \subseteq \mathbb{N}\times\mathbb{N}$. Further, B is infinite. So by property (ii), B is countably infinite, i.e., $\mathbb{N}\simeq B$. But then since $\mathbb{N}\simeq B$ and $B\simeq\mathbb{Q}^+$, we have that $\mathbb{Q}^+$ is countably infinite by (i).

(e) Examples of uncountably infinite sets include $\mathbb{R}$, $(0,1)\subseteq\mathbb{R}$, the set of functions $\mathbb{N}\to\{0,1\}$ (equivalently, the set of arbitrary-length bitstrings), and $\mathscr{P}(\mathbb{N})$.