Homework 2

Due: Wednesday, February 15

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

LaTeX tips

As you begin using LaTeX, here are some useful commands that might come in handy for this homework:

`\Pow`	$\mathcal{P}$	`\cap`	$\cap$	`\cup`	$\cup$
`\star`	$\star$	`\overline{A}`	$\overline{A}$	`\emptyset`	$\emptyset$
`\times`	$\times$	`\setminus`	$\setminus$	`\subseteq`	$\subseteq$
`\forall`	$\forall$	`\exists`	$\exists$	`\land`	$\land$
`\lor`	$\lor$	`\lnot`	$\lnot$	`\to`	$\to$

A note that LaTeX is *not* required until **Homework 3**.

Problem 1

Translate the sentences below into formulas of first-order logic. Use the following symbols. Note: the sets listed here can be used as *domains* of quantification. That is, you could write $\forall x : P, \ldots$ to quantify over all plants. You should not use any other domains of quantification.

- Sets:
  - P: the set of plants
- Predicates:
  - $G(x)$: "x is a geranium"
  - $L(x)$: "x is a lily"
  - $T(x, y)$ "x and y are planted together in the same pot"

  - Familiar mathematical symbols like $<$, $\leq$, $=$ have their normal meanings

- Functions:
  - $f(x)$: the number of flowers on x
  - $n(x)$: the plant nearest to x (that is not x itself)

- Constants:
  - g: Edith's favorite geranium

a. A geranium and a lily are planted in the same pot.

b. Some geranium has more flowers than every lily.

c. The plant nearest to Edith's favorite geranium is also a geranium, but is planted in a different pot.

d. No plant is both a geranium and a lily.

e. Explain your answer to part b. above. Suppose that you had to argue that this statement was true. How would you justify this claim? Think in terms of the proof rules that we have discussed in lecture.

Solution:

Students may come up with alternate translations. These are fine if they're logically equivalent.

a. $\exists x\ y : P, G(x) \land L(y) \land T(x, y)$

b. $\exists x : P, G(x) \land \forall y : P, L(y) \to f(x) > f(y)$

c. $G(n(g)) \land \neg T(g, n(g))$

d. $\neg \exists x : P, G(x) \land L(x)$ or $\forall x : P, \neg(G(x) \land L(x))$ or $\forall x : P, \neg G(x) \lor \neg L(x)$

e. Read out loud, this translation says something like "there is a plant x that is a geranium, and for every plant y, if y is a lily then x has more flowers than y." If I had to justify this claim, I would first provide a witness for the existential: I'd point at some specific geranium. This is an existential introduction. (Students may mention "and introduction" after this, which is fine, but not strictly required.) Then I'd have to show the universal claim. I can do this by taking an arbitrary plant y (forall introduction) and supposing it is a lily (implication introduction), then arguing that the geranium I pointed out has more flowers than that lily.

Problem 2

Let $A = \{\varnothing, 3, \text{“plant”}\}$, $B = \{9, 7, 6\}$, $C = \{3, 9, 6\}$, and $D = \{10i \mid i \in \mathbb{Z}, i \in [1, 9]\}$.

Find the cardinalities of the following sets. You only need to justify b, i, and j.

a. A
b. $\{A, C, 3\}$
c. $B \cup C$
d. $A \cap C$
e. $C \setminus B$
f. $\mathcal{P}(B)$
g. $\mathcal{P}(\mathcal{P}(B))$
h. $B \times D$
i. $\mathcal{P}(A \times A)$
j. $\mathcal{P}(\mathcal{P}(A) \times D)$

Note: Your answer may be given as a power of 2.

Solution:

a. 3. A contains three elements: $\varnothing$, 3, and plant.

b. 3.

   The set contains the elements A, C, and 3. Even though A and C are sets, they are elements of the set in this case, so they only count as one element each, and their inner elements are not important.

c. 4.

   $B \cup C = \{9, 7, 6, 3\}$. The cardinality of a set is the number of distinct elements, so 9 and 6 are not counted twice.

d. 1.

   The only element in both sets is 3.

e. 1.

   The only element in C but not B is 3.

f. 8.

   The cardinality of B is 3, and the cardinality of a power set is 2 to the power of the cardinality of the set, so $2^3 = 8$

g. 256.

The cardinality of $\mathcal{P}(B)$ is 8, and $2^8 = 256$

h. 27.

The Cartesian product of two sets is formed by pairing each element of the first with each element of the second, so its cardinality is equal to the product of the cardinalities of the two sets. $|B| \cdot |D| = 3 \cdot 9 = 27$

i. $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^9 = 512$

j. 2^{72}.

We apply the facts that the cardinality of a power set is 2 to the power of the cardinality of the set and that the cardinality of the Cartesian product is the product of the cardinalities of the two sets.

$2^{2^{|A|} \cdot |D|} = 2^{2^3 \cdot 9} = 2^{72}$

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Hw2.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, copy the entire file to your computer, and upload it to Gradescope.

Question 3 Autograded on Lean/Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Recall that we can create sets with *descriptions* using set-builder notation, like

$$A = \{x \in \mathbb{R} \mid x^2 > 1\}.$$

We would read this as "A is the set of x in the real numbers *such that* $x^2 > 1$." The condition that $x^2 > 1$ is a condition we place on this set to define it. Another example could be

$$T = \{t \mid t \text{ is a kind of tree}\}.$$

in which $T = \{\text{maple}, \text{elm}, \text{oak}, \dots\}$. This is an example of a description that is in natural language. One might consider if *every* description is a valid description.

Consider the description "X does not contain itself".

> **Example**
>
> For example, let
>
> $$T = \{t \mid t \text{ is a kind of tree}\}.$$
>
> be the set of all kinds of trees. This set T does not contain itself ($T \notin T$) since the set of all kinds of trees is not a kind of tree.
>
> However, let
>
> $$U = \{t \mid t \text{ is } \textit{not} \text{ a kind of tree}\}.$$
>
> be the set of everything that *isn't* a kind of tree. Well, U is a set of things, that clearly isn't a kind of tree, so $U \in U$ and we say U "contains itself".

Let's build a set with this specific description. Let S be the set that contains all sets that do not "contain themselves". That is, we define it to be

$$S = \{X \text{ is a set} \mid X \notin X\}$$

A set such as U *would not* be in S, as U contains itself. However, a set such as T *would* be in S, as T does not contain itself.

a. Show that the assumption that S is a member of S leads to a contradiction.

b. Show that the assumption that S is not a member of S leads to a contradiction.

c. What do these contradictions suggest about how we can or cannot define a set? This paradox is called *Russell's Paradox*. Are there any ways to resolve these contradictions in set theory? Do some research and cite at least one source.[1]

[1] This is an open ended question, and there is no right answer! You should demonstrate to us that you have an understanding of what is going on.

Solution:

$S = \{X \text{ is a set} | X \notin X\}$

a. If S is a member of itself, that would mean that $S \in S$ however, the set builder definition of the set S tells us that S contains all sets that do *not* contain itself. By definition, if S is a member of itself, it must not be a member of itself, which is a contradiction.

b. On the other hand, if S does not contain itself, that would mean that $S \notin S$, however, that would mean that, according to the set builder definition of S, it is a set that should be in the set S. Thus, by definition, if S is not a member of itself, then it must be a member of itself, which is a contradiction.

c. This theory suggests that a set cannot be determined by any well-defined condition (as initially assumed, according to Stanford's Plato Article). Thus, it tells us that there cannot exist a Set of all sets. Regarding solutions, it became clear that the way sets were constructed needed to change. The simple set theory, or naive set theory, is unrestricted. However, axiomatic set theory resolves these paradoxes. One response was by David Hilbert, who pushed to build an axiomatic foundation of mathematics. that would create a foundation for axiomatic set theory. Thus, not only well-defined and finite sets could be constructed, but also those that had certain rules of inference.
SOURCE: Irvine, Andrew David and Harry Deutsch, "Russell's Paradox", The Stanford Encyclopedia of Philosophy (Spring 2021 Edition), Edward N. Zalta (ed.),
URL = jhttps://plato.stanford.edu/archives/spr2021/entries/russell-paradox/i