Homework 10

Due: Wednesday, May 5th

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

LaTeX tips

Here are some LaTeX commands that might help you:

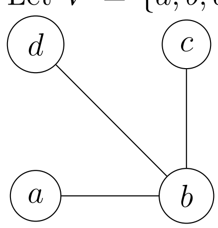
`\overline{G}` $\overline{G}$

You may draw your graphs by hand or any method you would like, and include images using `\includegraphics[scale=.5]{graph.png}`. If you are feeling brave, look into drawing graph diagrams in LaTeX using the `tikz` package!

Problem 1

Let $\mathbb{E}$ denote the set of edges of a complete graph with vertices V. Suppose G is a graph with vertex set $V(G) = V$ and edges $E(G) \subseteq \mathbb{E}$. The complement of G, $\overline{G}$, is the graph where $V(\overline{G}) = V$ and $E(\overline{G}) = \mathbb{E} \setminus E(G)$.

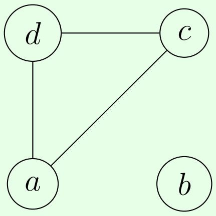
a. Let $V = \{a, b, c, d\}$. Draw the complement of the graph on V shown below.



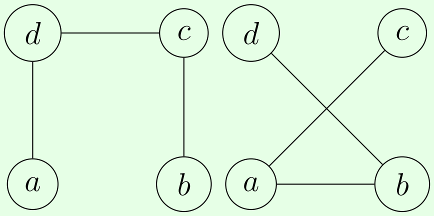
b. It is possible to have a graph G where both G and $\overline{G}$ are connected. Give an example of such a graph. Draw both G and $\overline{G}$. Recall, a graph is **connected** if there is a simple path between each pair of vertices (that is, all vertices are connected).

c. Suppose G is a graph on 9 vertices such that G has an Eulerian tour, and both G and $\overline{G}$ are connected. Prove that $\overline{G}$ must have an Eulerian tour.

Solution:

a. 

b. It is possible. As an example, consider the complementary graphs shown here:



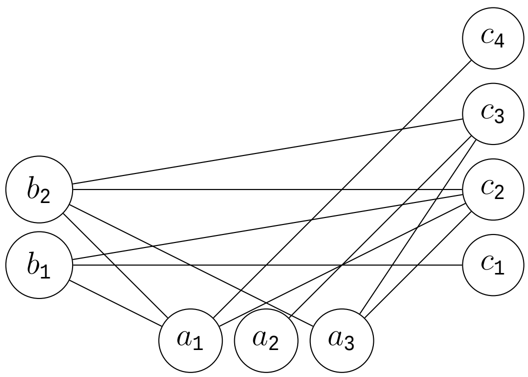
c. A connected graph has an Eulerian tour iff every vertex has even degree. Since G has an Eulerian tour, every vertex must have even degree. Since there are 9 vertices total, each vertex has 8 possible edges; each vertex must be adjacent to an even number of vertices and nonadjacent to an even number of vertices.

This means that in $\overline{G}$, each vertex is adjacent to an even number of vertices (since nonadjacencies in G are adjacencies in $\overline{G}$). So each vertex has even degree and $\overline{G}$ must have an Eulerian tour.

Problem 2

In this problem, we are interested in graphs G with the following structure: the vertex set $V(G)$ is partitioned into three nonempty disjoint sets $V(G) = V_1 \cup V_2 \cup V_3$, and edges only connect vertices in distinct partitions. If a graph G has this structure, we will call it *3-isolated*.

Here is an example of a 3-isolated graph with $V_1 = \{a_1, a_2, a_3\}, V_2 = \{b_1, b_2\}, V_3 = \{c_1, c_2, c_3, c_4\}$. Notice that there are no edges connecting any of the a vertices to each other, and similarly for the b and c vertices.



a. Consider the vertex set $V_1 = \{a_1, a_2, a_3\}, V_2 = \{b_1, b_2\}, V_3 = \{c_1, c_2, c_3, c_4\}$ shown above. How many possible 3-isolated graphs can be drawn on this vertex set?

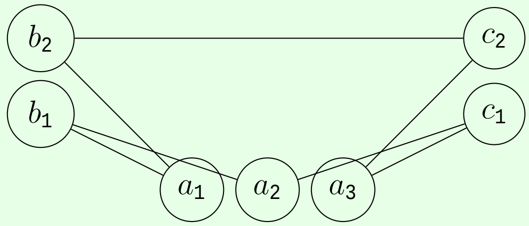
b. I make the following conjecture: on any vertex set V_1, V_2, V_3 with $|V_i| \geq 2$ for each i (in other words, each set of vertices has at least 2 vertices), it is impossible to draw a 3-isolated graph with a Hamiltonian cycle.

   Prove or disprove my conjecture!

Solution:

a. There are $3 * 2 = 6$ possible edges between V_1 and V_2, $3 * 4 = 12$ possible edges between V_1 and V_3, and $2 * 4 = 8$ possibilities between V_2 and V_3, for a total of 26 possible edges. We can choose to include each edge or not. So there are $2^{26} = 67108864$ possible graphs.

b. This conjecture is false. Here's a counterexample:


b_2
c_2
b_1
c_1
a_1
a_2
a_3