

Homework 9

Due: Wednesday, April 26

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Alan and En-Hua each have *weighted* coins: when flipped, these coins land on heads with probability $1/3$ and tails with probability $2/3$. Suppose Alan and En-Hua each toss their coin twice. Let X be the number of heads Alan obtains and Y the number of heads En-Hua obtains. In all of the problems below, show the calculation that leads to your answer.

- We want to model this situation as a probability space. Thinking in terms of sets: what is our *sample space* S ? What are the *outcomes*? What is the *event* $X = 1$?
- What is the distribution of X ? (In other words: for each value v in the range of X , what is $Pr[X = v]$?) What is the distribution of Y ?
- What is the distribution of $X + Y$?
- Given that $X + Y = 2$, find the distribution and expected value of X .

Problem 2

It is time for the annual Brown Horticultural Society's (BHS)¹ annual Great Garden Grow-off!²

The format of the Grow-off is as follows: 64 teams compete head-to-head in a single elimination tournament (with the healthiest-looking plants declared as winners). Thus, 32 games will be played in the first round, 16 in the second round, etc. with winners moving on to subsequent rounds, for a total of 63 games. There are no ties.

You have been tasked with predicting a winner for each of the 63 games *before the competition starts*. You receive 1 point for each correctly guessed winner for the first round, with the points rewarded for each correct prediction doubling for each subsequent round—you receive 32 points for correctly guessing the winner of the final championship match in the 6th round.

There's a catch: you are an absolute gardening novice with no ability to determine the quality of the plants whatsoever, thus you've left each of your predictions to a fair coin flip, i.e. for a given match-up you pick a predicted winner with 50/50 odds. How many points do you expect to earn?

Hint: What is the probability of correctly guessing the winner of a game in round 1? round 2? What about in round n ?

¹Fictionalized organization.

²Fictionalized competition.

Problem 3

You have a straight, 100cm length of vine, with thorns at each centimeter, from 0 to 100 inclusive. You dip it into liquid nitrogen and drop it. It breaks exactly halfway between two adjacent thorns, where the point of breakage is equally likely to be in between any pair of adjacent thorns. One piece must be longer than the other piece, since it couldn't have broken exactly at the middle thorn; call the shorter piece S and the longer piece L .

- a. What's the expected number of thorns on S ?

To make this clearer, if we were talking about a length of vine with ten evenly spaced thorns on it and it broke between thorns 7 and 8, there would be 7 thorns on the longer piece, and the remaining three thorns would be on the shorter piece.

- b. What about the expected number of thorns on L ?
- c. What is the expected value of the *product* of the number of thorns on S and the number of thorns on L ? (Recall that we proved some formulas for sums, e.g. for $\sum_{i=0}^n i^2$, using induction earlier this semester. You do not need to re-prove these formulas.)



Problem 4 (Mind Bender — *Extra Credit*)

This week's mindbender is a Lean question!

The problem can be found by navigating to `BrownCs22/Homework/Hw9.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.