Homework 5

Due: March 9, 2023

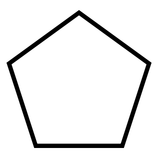
All homeworks are due at 11:59 PM on Gradescope.

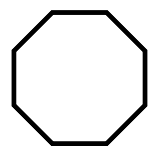
Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

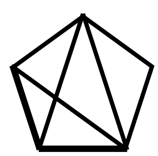
Problem 1

A *regular n-gon* is a polygon with n sides, all of equal length, and n angles, all of equal measure. For example, a square is a regular 4-gon, and the images below are a regular 5-gon and 8-gon:





A *diagonal* of an n-gon is a line connecting two non-adjacent vertices. For instance, here are three diagonals of the regular 5-gon:



Show using induction that for all $n \in \mathbb{N}$ where $n \geq 3$, a regular n-gon always has $\frac{(n)(n-3)}{2}$ diagonals.

Problem 2

The TAs are purchasing some plants to decorate the front of the DeCiccio Auditorium stage during CS 22 lectures. The front of the stage can hold up to 100 pounds; the TAs, being floral aficionados, would like to make maximal use of this carrying capacity.

The CS 22 Plant Emporium sells two types of potted plants: bulbs, which each weigh 8 pounds; and succulents, which each weigh 18 pounds.

a. Determine how many of each plant the TAs should purchase if they would like to maximize the total *weight* of the plants they buy. (Remember that they cannot exceed the carrying capacity of the stage.) Justify your answer.

b. But wait, the CS 22 Plant Emporium has an innovative new product that defies the laws of physics: the *anti-plant*(™)! For every plant species the Emporium sells, they now also sell an *anti-species* that has a weight equal to the *negative* of that of the original plant (e.g., an anti-bulb weighs -8 pounds, and an anti-succulent weighs -18 pounds).

   Now that the TAs can purchase anti-plants as well as traditional ones, describe all possible configurations of plants (including anti-plants) they can buy if they want their purchase to weigh exactly 100 pounds. Prove that these configurations all weigh exactly 100 pounds, and that they are the only possible solutions.

   Example

   If succulents weighed 97 pounds and bulbs weighed 3 pounds, we could describe all possible configurations as follows: for any $x \in \mathbb{Z}$, we purchase $1+3x$ succulents (or anti-succulents if this number is negative) and $1-97x$ bulbs (or anti-bulbs).

You may cite without proof the following lemma: for all $a, b, c \in \mathbb{Z}$, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$

Problem 3

The Fibonacci numbers are defined recursively, according to the following rules:

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 1 \\
f_{n+2} &= f_n + f_{n+1}
\end{aligned}
$$

Example

$f_2 = f_0 + f_1 = 2$. $f_3 = f_1 + f_2 = 3$. $f_4 = f_2 + f_3 = 5$.

Prove using induction that for every n, f_n and f_{n+1} are relatively prime numbers.

You will likely find it useful to prove that for all $a, b \in \mathbb{Z}$, $\gcd(a, b + a) = \gcd(a, b)$.

Problem 4 (Mind Bender — *Extra Credit*)

Let $\langle a_k \rangle_{k \in \mathbb{N}}$ be the sequence of natural numbers defined as follows:

- $a_0 = 0$.
- $a_1 = 1$.
- For all natural $k \geq 2$, $a_k = 2a_{k-1} + a_{k-2}$.

a. Show that for all $n \in \mathbb{N}$ such that $n \geq 1$, we have $\sum_{k=0}^{n} a_k < 2a_n$.

b. Show that we can write any number $n \in \mathbb{N}$ as the sum/difference of elements of the sequence, i.e., $n = a_{j_1} \pm a_{j_2} \pm \cdots \pm a_{j_r}$ for some indices $j_1, j_2, \ldots, j_r$.

   (More formally, we are asking you to prove that for any $n \in \mathbb{N}$, there exist some

   - number of terms $r \in \mathbb{N}$,
   - indices $\{j_s \mid s \in \mathbb{N} \text{ and } s < r \text{ and } \forall s \in \mathbb{N}, j_s \in \mathbb{N}\}$, and
   - exponents $\{p_s \in \{0, 1\} \mid s \in \mathbb{N} \text{ and } s < r\}$

   for which $n = \sum_{s=0}^{r-1} (-1)^{p_s} a_{j_s}$.)

   You may cite without proof the fact that the sequence is strictly increasing for $k \geq 1$. You may also make use of your result in the preceding part.

Before trying to prove this, pick some particular values of n and try to write them as sums/differences of the terms of the sequence, and look for patterns as you do so. (You might find it useful to pick consecutive values of n.) In particular, consider whether each natural number n is less than, greater than, or equal to the sum of all sequence elements strictly less than it. How does this relate to whether you need to add terms or subtract them to obtain n?