Homework 3

Due: Feb 24, 2023

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

LaTeX tips

Here are some more LaTeX commands that might help you

`\subseteq` $\subseteq$ `\sqrt{a}` $\sqrt{a}$ `\simeq` $\simeq$
`\{a\mid a\in\Z}` $\{a \mid a \in \mathbb{Z}\}$

LaTeX *is* required starting with this homework. You should be using the `cs22` document class. Look for a LaTeX guide here.

Problem 1

For each proposition below, determine whether it is true for arbitrary sets A and B. If it is true, prove it! If it is false, provide a counterexample: that is, give two sets A and B for which the claim is false, and explain why it is false.

a. $|A \cup B| = |A| + |B|$

b. $|A \setminus B| = |A| - |B|$

c. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

Problem 2

When there is a *bijection* between two finite sets, we can conclude that those sets have the same number of elements. (The mindbender below explores what happens with infinite sets.)

Let A be a set with n elements. Let T be the set of all ordered pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Define a bijection between T and S (and prove that it is a bijection).

Conclude that T and S must be the same size.

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Hw2.lean` in the directory browser on the left of your screen in Gitpod. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, copy the entire file to your computer, and upload it to Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

> *Mind Benders* are extra credit problems intended to be more challenging than usual homework problems and are an exploration into a topic not covered in lecture. This week, we have an exploration into a concept called *equinumerosity* and *countability*.

Thought you learned how to count years ago? Think again!

This question explores how we can use the tools we've learnt, such as functions, to define a more general notion of counting. Especially in the case of counting *infinitely* many things.

Let A and B be sets, not necessarily finite. We say that A and B are *equinumerous* if there exists a bijective function $f : A \to B$. We use the notation $A \simeq B$ to mean "A is equinumerous to B."

> **Example**
>
> $\{a, b, c, d\}$ is equinumerous to $\{0, 1, 2, 3\}$ (what function f can we find?). $\{a, b, c, d\}$ has four elements.
>
> In general, A has n elements if $A \simeq \{0, \dots, n-1\}$. We say that a set is *finite* if it has n elements, for some $n \in \mathbb{N}$.

But what about infinite sets? We say that A is *countably infinite* if $\mathbb{N} \simeq A$ (that is, we can find a bijective function $f : \mathbb{N} \to A$)[1]. But equinumerousity of infinite sets can be confusing!

a. Show that $\mathbb{N}^+$ (the set $\{1, 2, \dots\}$) is countably infinite: that is, show that there is a bijective function $f : \mathbb{N} \to \mathbb{N}^+$.

   This is not immediately obvious, since $\mathbb{N}^+ \subset \mathbb{N}$! That is, $\mathbb{N}$ contains 1 more element than $\mathbb{N}^+$ but they are still equinumerous!

b. Is it true that $\mathbb{Z}$ is countably infinite? Why or why not? If it is, prove so (that is, find a bijection $f : \mathbb{N} \to \mathbb{Z}$). If it isn't, prove such a bijection doesn't exist.

[1] Intuitively, this makes a bit of sense. We think of $\mathbb{N}$ to be the counting numbers, so if we can attach a 'count' to each element in A, even if A is infinite, it is 'countable' in some sense

c. Show that $\mathbb{N} \times \mathbb{N}$, the set of pairs of natural numbers, is countably infinite.

> **Hint:** Writing down an explicit bijection f is hard. You can draw a picture here, or explain precisely how you would "count" all of the pairs.

d. We give you these two facts, that you do not need to prove yourself:

   i. The relation $\simeq$ is transitive: if $A \simeq B$ and $B \simeq C$, then $A \simeq C$.

   ii. If $A \subseteq B$, A is not finite, and B is countably infinite, then A is countably infinite.

   Using the above two facts and part c., show that the set of positive rational numbers $\mathbb{Q}^+$ is countably infinite. Can you also conclude that $\mathbb{Q}$ is also countably infinite?

e. It's starting to sound like a lot of sets are countably infinite! Can you think of an infinite set that is *not* countable? You don't need to prove that it it is uncountable, but indicate why you think it isn't.

The following aren't for credit, but are some extra food for thought:

f. What set did you come up with? Can you prove that it is somehow not countably infinite?

g. You might wonder whether *all* uncountably infinite sets are equinumerous? Is there anything *even bigger* than uncountable infinity? Can you think of two uncountably infinite sets that aren't equinumerous. How would you prove so?

> You might want to look up *diagonalization* and *Cantor's Theorem.*