CSCI0220 Practice Midterm

Please read in full:

The following practice exam will look somewhat different from the one you receive on exam day. This practice exam is derived from last year's midterm.

The in-class exam this year will cover similar information and be of similar difficulty in questions, but will be designed and formatted for an in-class exam. *You should not draw any conclusions about the length of the exam from this practice exam.*

It's a good idea to try the exam first by yourself, and then read over the solutions.

Problem 1

a. Prove using the set-element method that for sets Y, B, and C with the same universal set U:

$$(Y \setminus B) \cap (Y \setminus C) = Y \setminus (B \cup C).$$

Solution:

First we show that $(Y \setminus B) \cap (Y \setminus C) \subseteq Y \setminus (B \cup C)$.

Suppose $x \in (Y \setminus B) \cap (Y \setminus C)$.

Then $x \in Y \setminus B$ and $x \in Y \setminus C$.

So $x \in Y$ but $x \notin B$, and $x \in Y$ but $x \notin C$.

In other words, $x \in Y$ and $x \notin B$ and $x \notin C$.

By DeMorgan's law, $x \notin B \cup C$, so $x \in Y \setminus (B \cup C)$.

Now we show $Y \setminus (B \cup C) \subseteq (Y \setminus B) \cap (Y \setminus C)$.

Suppose $x \in Y \setminus (B \cup C)$.

Then $x \in A$ but $x \notin (B \cup C)$.

By DeMorgan's law, $x \notin B$ and $x \notin C$.

So $x \in Y$ but $x \notin B$, and $x \in Y$ but $x \notin C$.

So $x \in (Y \setminus B) \cap (Y \setminus C)$.

b. The following proposition is false.

Not a theorem. For any sets A, B, C, and D, define

$$L = (A \cup B) \times (C \cup D)$$
$$R = (A \times C) \cup (B \times D)$$

Then $L = R$.

Show a counterexample to this proposition: Give four sets A, B, C, D for which this equality is false. Explicitly give an example of an element that is in either L or R but is not in the other.

Solution:

Define: $A = \{a\}, B = C = \emptyset, D = \{b\}$.

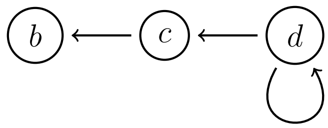
Then $L = (\{a\} \cup \emptyset) \times (\emptyset \cup \{b\}) = \{a\} \times \{b\} = \{(a, b)\}$.

And $R = (\{a\} \times \emptyset) \cup (\emptyset \times \{b\}) = \emptyset \cup \emptyset = \emptyset$. Note that for any set S, $\emptyset \times S = \emptyset$, since there are no pairs (x, y) with $x \in \emptyset$.

So $(a, b) \in L$ but $(a, b) \notin R$.

Problem 2

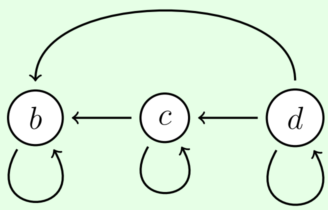
a. This diagram shows a relation R on the set $\{b, c, d\}$, where xRy if and only if there is an arrow from x to y.



By *adding* arrows to this diagram, without removing any existing arrows, turn the relation R into one that is both reflexive and transitive.

Is your relation also symmetric? Explain why or why not!

Solution:



This relation is *not* symmetric. Symmetric means that each arrow points in both directions. In this picture, we have c R b but not b R c.

b. We now consider the relation $\mathsf{is\_gcd} : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}^+$. This is a relation between pairs of integers and positive integers. $\mathsf{is\_gcd}((a, b), g)$ holds exactly when g is the greatest common divisor of a and b.

  i. Is is_gcd an injective relation? Is it surjective? For each of your answers, give a proof or a counterexample.

  ii. The Euclidean algorithm can be said to "compute" the is_gcd relation. Given input (a, b) the algorithm should produce a g such that $\mathsf{is\_gcd}((a, b), g)$ holds. Showing the intermediate steps, use the Euclidean algorithm to compute the gcd of 68 and 28.
  (Note: you do *not* need to use the extended Euclidean algorithm.)

Solution:

  i. It is not injective: $\mathsf{is\_gcd}((5, 5), 5)$ and $\mathsf{is\_gcd}((5, 10), 5)$ both hold. Two elements of the domain that are related to the same element of the codomain.

  It is surjective: suppose $g \in \mathbb{Z}^+$. Since g is not 0 ($0 \notin \mathbb{Z}^+$), we know that g must be the greatest common divisor of g and g. So $\mathsf{is\_gcd}((g, g), g)$

holds. For every element of the codomain, we have found an element of the domain related to it.

ii.

$$\begin{aligned} gcd(28, 68) &= gcd(12, 28) \\ &= gcd(4, 12) \\ &= gcd(0, 4) \\ &= 4 \end{aligned}$$

Problem 3

Tyler, Ryan, and Joseph are bank robbers who are very precise about the amount of money they steal: Each robbery, they steal precisely $n^3 - n$ dollars for some natural number n.

Use induction to prove that they are always able to divide the money they steal evenly among the three of them.

(Note: even if you are unable to finish the problem, show how you would set up your proof!)

Solution:

Translating this to mathematics, we want to show that for every natural number n, $3 \mid n^3 - n$. We do so by induction with predicate $P(n) = 3 \mid n^3 - n$.

Base case: $P(0)$ says $3 \mid 0^3 - 0$. It is true that $3 \mid 0$!

Inductive case: Suppose $3 \mid k^3 - k$. We want to show $3 \mid (k+1)^3 - (k+1)$.

Via algebra: $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k + 1 = (k^3 - k) + (3k^2 + 3k) = (k^3 - k) + 3(k^2 + k)$.

By our induction hypothesis, $3 \mid (k^3 - k)$, and $3 \mid 3(k^2 + k)$, so 3 divides their sum. We have shown our target.

So by induction, it holds that $3 \mid n^3 - n$ for every n.

Problem 4

a. Let x be a propositional formula with Boolean inputs a, b and c and the following truth table:

a	b	c	x
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Write a formula in disjunctive normal form that is equivalent to the formula x. Explain how you arrived at your answer.

Solution:

Let $x = (a \land b \land \neg c) \lor (\neg a \land b \land \neg c)$.

The first clause is true only when we are in the first T row of the table. The second clause is true only when we are in the second T row of the table. Otherwise, our formula is false. So it has exactly the same truth table as the one displayed.

b. Is the formula $\neg q \to ((p \lor q) \to p)$ unsatisfiable, satisfiable (but not valid), or valid? Again, explain your answer!

Solution:

The formula is valid: it is true under every truth assignment. We can see this by drawing out the truth table.

