Final Exam Reference Sheet

This sheet contains helpful properties and definitions to be used throughout the exam. You do not need to cite any rules found here throughout your work. This is not comprehensive.

Logic

- **Disjunctive Normal Form:** a disjunction of conjunctions. EX: $(P \wedge Q) \vee (S \wedge R)$
- **Conjunctive Normal Form:** a conjunction of disjunctions. EX: $(P \vee Q) \wedge (S \vee R)$

Set Identities

1. **Set Difference Law:** For all sets A and B, $A \setminus B = A \cap \overline{B}$
2. **Distributive Law:** For all sets A, B, and C
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. **Identity Law:** For all sets A,
$$A \cup \emptyset = A, A \cap U = A$$
4. **De Morgan's Law:** For all sets A and B,
$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Relations and Functions

1. A **bijective** function is one that is **injective** and **surjective**.
   (a) A function $f : A \to B$ is **injective** if, for all $a, b \in A$, $f(a) = f(b) \to a = b$.
   (b) A function $f : A \to B$ is **surjective** if, for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Number Theory

Definition 1: We say that a *divides* b, denoted $a \mid b$, when $b = ka$ for some $k \in \mathbb{Z}$.

Definition 2: We say that a is *congruent* to b mod m, denoted $a \equiv b \pmod{m}$, if $m \mid (b - a)$.

Properties of Congruence Relations:

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$, $a + c \equiv b + c \pmod{m}$ for any $c \in \mathbb{Z}$.

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$, $a \times c \equiv b \times c \pmod{m}$ for any $c \in \mathbb{Z}$.

Counting

The **binomial coefficient**, also called n *choose* k, is defined to be, for $n \geq k$ and $n, k \in \mathbb{N}$,

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Stars and Bars The number of ways to distribute m identical objects among n distinct groups is

$$\binom{m+n-1}{n-1}.$$

Probability

Inclusion-Exclusion: For events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

The **conditional probability** $\Pr(A|B)$ is the probability that A happened given that we know B did. It is defined as

$$\Pr(A|B) \stackrel{\text{def}}{=} \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A) * \Pr(A)}{\Pr(B)} \quad \text{(Bayes' Rule)}$$

A is **independent** of B if $\Pr(A|B) = \Pr(A)$, or if $\Pr(B) = 0$.

The **expected value** (or just expectation) of a random variable is a probability-weighted average of its values. The expected value of a random variable R is:

$$\mathbb{E}[R] = \sum_{\omega \in S} R(\omega) \Pr(\omega)$$

Graph Theory

If $u, v \in V(G)$, u is **adjacent to** v if $\{u, v\} \in E(G)$.

Eulerian Tour: a cycle which uses every edge in a graph exactly once. The length of an Eulerian tour is $|E(G)|$.

Hamiltonian Tour: a cycle that visits every vertex exactly once (except the start/end vertex). The length of a Hamiltonian tour is $|V(G)|$.