Proofs About Sets & Quantification

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CS 0220 2024

February 12, 2024
Overview

1. Proofs about Sets, Proof by Cases (1.7)

2. Predicate Formulas (3.6)
Set equality

When are two sets equal?

If $A$ and $B$ are sets, $A = B$ if and only if $\forall x, x \in A \leftrightarrow x \in B$.

Equivalently: $(\forall x, x \in A \rightarrow x \in B) \land (\forall x, x \in B \rightarrow x \in A)$.

Equivalently: $A \subseteq B \land B \subseteq A$.

This suggests a proof technique for proving set equality: the set-element method.
Proving set equalities: set-element method

Theorem: For any sets $A$ and $B$ of elements in universe $U$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(DeMorgan’s Law again! Now, connects intersection and union instead of $\land$ and $\lor$.)

Proof. We proceed by the set element method. First, we show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. This means that $x$ is not in both $A$ and $B$. Written as a formula: $\neg(x \in A \land x \in B)$. By De Morgan’s law, this is equivalent to $\neg(x \in A) \lor \neg(x \in B)$. So $x \in \overline{A} \lor x \in \overline{B}$, as desired.

Now we show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$. This means that $x \in \overline{A} \lor x \in \overline{B}$. Equivalently, $\neg(x \in A) \lor \neg(x \in B)$. By De Morgan again, this is equivalent to $\neg(x \in A \land x \in B)$, so $x \in \overline{A \cap B}$ as desired.
Fact about groups of people

Any two people have either met or not.

Given a set of people $G$, if all pairs of people in $G$ have met, we’ll call it a $club$. If no two people in $G$ have met, we’ll call them $strangers$.

**Theorem.** Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Does that seem true? Try some examples on the board.
Proof (Part 1)

The proof is by case analysis. Let $x$ denote one of the six people. Let $R = G \setminus \{x\}$ be the rest. There are two cases:

1. Among $R$, at least 3 have met $x$.
2. Among $R$, at least 3 have not met $x$.

At least one of these cases must hold. Since $|R|$ is odd, either more than half in $R$ know $x$ or less than half in $R$ know $x$ (and therefore more than half do not know $x$).

Case 1: At least 3 have met $x$. Let $J \subseteq R$ be those individuals. Two subcases:

1.1 No pair in $J$ have met each other. So, $J$ is a group of at least 3 strangers and the theorem holds in this subcase.
1.2 Some pair in $J$ have met each other. That pair and $x$ are a club of 3 people and the theorem holds in this subcase, too.

That covers Case 1!
Case 2: At least 3 have not met $x$. Let $J \subseteq R$ be those individuals. Two subcases:

2.1 Every pair in $J$ have met each other. So, $R$ is a club of at least size 3 and the theorem holds in this subcase.

2.2 Some pair in $J$ haven’t met each other. That pair and $x$ are a group of strangers of 3 people and the theorem holds in this subcase, too.

That covers Case 2! It’s kind of the inverse-video version of Case 1.

Since we showed that only these two cases can occur and the theorem holds in both, the theorem *always* holds.
Mixing quantifiers

**Theorem** (sparse squares): There’s a perfect square arbitrarily far from its closest perfect square.

Clear? Maybe a tad vague. True? How do we say this in logic?

\[ \forall d : \mathbb{N}, \exists i : \mathbb{N}, \forall j : \mathbb{N}, \]
\[ (i \text{ is a perfect square}) \land (|i - j| \leq d \rightarrow \neg(j \text{ is a perfect square})). \]

The expressions nest inside each other. The order matters.

You can think of it like a little game. I’m claiming that you can pick any \(d\) you want. I’ll then pick an \(i\) that’s a perfect square AND no matter what \(j\) you pick that is within \(d\) values of \(i\), \(j\) won’t be a perfect square.

So, what’s my winning strategy?
Any ambiguity is too many

“If you can identify any bird, you’ve got talent.”
1. If ∃b, you can identify b, then you’ve got talent.
2. If ∀b, you can identify b, then you’ve got talent.

“...statistics show that, in the UK, someone brews a cup of tea every second.”
1. ∀t, ∃p, p brews a cup of tea at second t
   “That person’s name is Nigel.”
2. ∃p, ∀t, p brews a cup of tea at second t
DeMorgan returns: Negating quantifiers

These two statements are equivalent:

■ Not everyone likes coffee.
■ There’s someone who doesn’t like coffee.

\( \neg \forall x, P(x) \) is equivalent to \( \exists x, \neg P(x) \).