Proofs About Sets & Quantification

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Overview

Set equality

When are two sets equal?

If A and B are sets, $A = B$ if and only if $\forall x, x \in A \leftrightarrow x \in B$.

Equivalently: $(\forall x, x \in A \rightarrow x \in B) \wedge (\forall x, x \in B \rightarrow x \in A)$.

Equivalently: $A \subseteq B \wedge B \subseteq A$.

This suggests a proof technique for proving set equality: the *set-element method*.

Proving set equalities: set-element method

Theorem: For any sets A and B of elements in universe U, $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(DeMorgan's Law again! Now, connects intersection and union instead of $\wedge$ and $\vee$.)

Proof. We proceed by the set element method. First, we show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. This means that x is *not* in both A and B. Written as a formula: $\neg(x \in A \wedge x \in B)$. By De Morgan's law, this is equivalent to $\neg(x \in A) \vee \neg(x \in B)$. So $x \in \overline{A} \vee x \in \overline{B}$, as desired.

Now we show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$. This means that $x \in \overline{A} \vee x \in \overline{B}$. Equivalently, $\neg(x \in A) \vee \neg(x \in B)$. By De Morgan again, this is equivalent to $\neg(x \in A \wedge x \in B)$, so $x \in \overline{A \cap B}$ as desired.

Fact about groups of people

Any two people have either met or not.

Given a set of people G, if all pairs of people in G have met, we'll call it a *club*. If no two people in G have met, we'll call them *strangers*.

Theorem. Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Does that seem true? Try some examples on the board.

Proof (Part 1)

The proof is by case analysis. Let x denote one of the six people. Let $R = G \setminus \{x\}$ be the rest. There are two cases:

1. Among R, at least 3 have met x.
2. Among R, at least 3 have *not* met x.

At least one of these cases must hold. Since $|R|$ is odd, either more than half in R know x or less than half in R know x (and therefore more than half do not know x).

Case 1: At least 3 have met x. Let $J \subseteq R$ be those individuals. Two subcases:

1.1 No pair in J have met each other. So, J is a group of at least 3 strangers and the theorem holds in this subcase.

1.2 Some pair in J have met each other. That pair and x are a club of 3 people and the theorem holds in this subcase, too.

That covers Case 1!

Proof (Part 2)

Case 2: At least 3 have not met x. Let $J \subseteq R$ be those individuals. Two subcases:

2.1 Every pair in J have met each other. So, R is a club of at least size 3 and the theorem holds in this subcase.

2.2 Some pair in J haven't met each other. That pair and x are a group of strangers of 3 people and the theorem holds in this subcase, too.

That covers Case 2! It's kind of the inverse-video version of Case 1.

Since we showed that only these two cases can occur and the theorem holds in both, the theorem *always* holds.

Mixing quantifiers

Theorem (sparse squares): There's a perfect square arbitrarily far from its closest perfect square.

Clear? Maybe a tad vague. True? How do we say this in logic?

$$\forall d : \mathbb{N}, \exists i : \mathbb{N}, \forall j : \mathbb{N},$$
$$(i \text{ is a perfect square}) \land (|i - j| \le d \to \neg(j \text{ is a perfect square})).$$

The expressions nest inside each other. The order matters.

You can think of it like a little game. I'm claiming that you can pick any d you want. I'll then pick an i that's a perfect square AND no matter what j you pick that is within d values of i, j won't be a perfect square.

So, what's my winning strategy?

Any ambiguity is too many

"If you can identify any bird, you've got talent."

1. If $\exists b$, you can identify b, then you've got talent.
2. If $\forall b$, you can identify b, then you've got talent.

"...statistics show that, in the UK, someone brews a cup of tea every second."

1. $\forall t, \exists p$, p brews a cup of tea at second t

   "That person's name is Nigel."

2. $\exists p, \forall t$, p brews a cup of tea at second t

DeMorgan returns: Negating quantifiers

These two statements are equivalent:

- Not everyone likes coffee.
- There's someone who doesn't like coffee.

$\neg\forall x, P(x)$ is equivalent to $\exists x, \neg P(x)$.