The Language of Set Theory

Robert Y. Lewis

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Overview

Mathematical languages

We're building up a *formal language* for talking about propositions.

Natural language is confusing and ambiguous. Ours is not. (Fingers crossed!)

A new part of our language today: *sets*. Are we *defining* sets? Or introducing them as an *atomic concept*?

Either way! Really useful *vocabulary* for talking about things, mathematical and otherwise.

Set Definition

Definition (informal): A *set* is a bunch/collection/group of objects.

Definition: The *elements* of the set are the objects contained in that set.

Sets can contain numbers, ordered sequences of numbers, strings, names, or other sets.

Objects are either *in* the set or *not in* the set. We don't have a concept of an object being in a set multiple times. It's a Boolean property.

We write curly braces around a comma-separated list to build a set.

Examples:

- $H = \{$ Allie, Carmen, Jania, Joseph, Tyler $\}$
- $I = \{$ this computer, this slide clicker, that projector screen $\}$
- $J = \{$ "this computer", "this slide clicker", "that projector screen" $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

Elements

- $H = \{$ Allie, Carmen, Jania, Joseph, Tyler $\}$
- $I = \{$ this computer, this slide clicker, that projector screen $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

Definition: We say say $x \in S$ if x is *an element of* or *in* or *a member of* the set S.

- Tyler $\in H$?
- this computer $\in H$?
- this computer $\in I$?
- Jania $\in \mathbb{N}$?

- Yes.
- No. This computer $\notin H$.
- Yes.
- No. Jania $\notin \mathbb{N}$.

Sets of sets

- $A = \{1, 4, 9\}$
- $B = \{\{1, \{4\}\}, \{9\}\}$

- $1 \in A$?
- $1 \in B$?
- $\exists x, x \in B \land 1 \in x$?

- Yes.
- No, but $\{1, \{4\}\} \in B$.
- Yes, $x = \{1, \{4\}\} \in B$ and $1 \in x$.

Some Sets of Numbers

- $\emptyset = \{\}$ (empty set, null set)
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
- $\mathbb{Q} = \{1/2, -4/15, 21, \ldots\}$ (rationals)
- $\mathbb{R} = \{\sqrt{2}, -\pi, 21, \ldots\}$ (real numbers)
- $\mathbb{C} = \{i/2, 15 - i, \sqrt{7}, 21, \ldots\}$ (complex numbers)

Superscript plus limits to (strictly!) positive values: $\mathbb{Z}^+ = \mathbb{N}^+$.

Superscript minus limits to negative values: $21 \notin \mathbb{R}^-$.

Subsets

Definition: One set is a *subset* of another if every element of the first set is also an element of the second.

We write $S \subseteq T$ to say the set S is a subset of set T. So, $S \subseteq T$ means $\forall x \in S, x \in T$. Could also write $\forall x, x \in S \to x \in T$.

Examples:

- $\mathbb{N} \subseteq \mathbb{Z}$? Yes, every positive integer is also an integer.
- $\mathbb{Z}^+ \subseteq \mathbb{N}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{C} \subseteq \mathbb{Z}$? No, $\mathbb{C} \not\subseteq \mathbb{Z}$. Some (many!) complex numbers are not integers. Although, $\mathbb{Z} \subseteq \mathbb{C}$.
- $\mathbb{N} \subseteq \mathbb{N}$. Yes, if sets are equal, all of the first must also be in the second!

Note: $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ looks a little bit like $3 \leq 4$.
We write $A \subset B$ to rule out equality (like $a < b$).

Operations on sets: Union

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *union* of sets X and Y, $X \cup Y$, consists of every element that is in either X or Y. In other words, $z \in X \cup Y$ means $z \in X \vee z \in Y$.

Example: $B \cup C = \{j, a, n, i, l, e\}$. Order doesn't matter: = $\{a, e, i, j, l, n\}$

Operations on sets: Intersection

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *intersection* of sets X and Y, $X \cap Y$, consists of every element that is in both X and Y. In other words, $z \in X \cap Y$ means $z \in X \wedge z \in Y$.

Example: $A \cap E = \{e\}$. $B \cap D = \emptyset$

Operations on sets: Set difference

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *set difference* of sets X and Y, $X \setminus Y$, consists of every element that is in X but not in Y. In other words, $z \in X \setminus Y$ means $z \in X \land z \notin Y$.

Example: $C \setminus B = \{l, e\}$.

Example: $E \setminus D = \{c, a, m, n\}$.

Operations on sets: Symmetric difference

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *symmetric difference* of sets X and Y, $X \triangle Y$, consists of every element that is in X but not in Y or in Y but not X. In other words, $z \in X \triangle Y$ means $(z \in X \wedge z \notin Y) \vee (z \in Y \wedge z \notin X)$. That is, $z \in X$ XOR $z \in Y$.

Example: $A \triangle B = \{o, s, e, p, h, a, n, i\}$.

Example: $C \triangle D = \{a, i, r, t, y\}$. (Remember: order doesn't matter)

Operations on sets: Complement

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *complement* of a set X, $\overline{X}$, is defined with respect to some universe of possible elements U. It consists of every possible element that is not in X. In other words, $\overline{X} = U \setminus X$.

Example: If U is the universe of all letters in English, $\overline{A} = \{a, b, c, d, f, g, i, k, l, m, n, q, r, t, u, v, w, x, y, z\}$.

Example: If $U = \mathbb{Z}$, $\mathbb{Z}^- = \overline{\mathbb{Z}^+} \setminus \{0\}$.

Disjoint sets

Definition: Sets X and Y are *disjoint* if they have no elements in common.

$X \cap Y = \emptyset$ or

$X \subseteq \overline{Y}$.

Operations on sets: Power set

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *power set* of a set X, $\mathcal{P}(X)$, is the set of all subsets of X. In other words, $\forall x \in \mathcal{P}(X), x \subseteq X$ and $\forall x \subseteq X, x \in \mathcal{P}(X)$.

Example: $\mathcal{P}(\{r, o, b\}) = \{\{\}, \{r\}, \{o\}, \{b\}, \{r, o\}, \{r, b\}, \{o, b\}, \{r, o, b\}\}$.

Example: $\mathcal{P}(B) = \{\{\}, \{j\}, \{a\}, \{n\}, \{i\}, \{j, a\}, \{j, n\}, \{j, i\}, \{a, n\}, \ldots, \{j, a, n, i\}\}$.

Example: $\mathcal{P}(\emptyset) = \{\emptyset\}$.

Operations on sets: Cardinality

- $A = \{j, o, s, e, p, h\}$
- $B = \{j, a, n, i\}$
- $C = \{a, l, i, e\}$
- $D = \{t, y, l, e, r\}$
- $E = \{c, a, r, m, e, n\}$

Definition: The *cardinality* of a set X, $|X|$, is the count of the number of (unique) elements in X.

Example: $|A| = 6, |B| = 4, |C| = 4, |D| = 5, E = 6$

Example: $|\emptyset| = 0$.

Example: If $|A| = n$, $|\mathcal{P}(A)| = 2^n$. Each subset consists of a decision of whether to include or not include (2 possibilities) each of the n elements of A.

Building sets with predicates

General form: $\{$ description of a set $|$ filter on the set $\}$.

Examples:

- $A = \{n \in \mathbb{N} \mid n = 2k + 1 \text{ for some integer } k\}$
- $B = \{x \in \mathbb{R} \mid x^2 > 1\}$

Note: Python has a notation for this idea.

Products of sets

- $C = \{2, 5\}$
- $D = \{a, b, c\}$

- $C \times D = \{(2, a), (2, b), (2, c), (5, a), (5, b), (5, c)\}$
- $\mathbb{N} \times D = \{(0, a), (0, b), (0, c), (1, a), (1, b), \ldots\}$
- $\mathbb{N} \times \mathbb{N} =$ the set of *ordered pairs* of natural numbers

Ordered pair: $(2, 0)$ is not the same as $(0, 2)$!

Concept check

- $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$, the even natural numbers
- $T = \{n \in \mathbb{N} \mid n < 10\}$
- $U = \{1, 2, 3\}$

What are the following sets?

- $E \cap T$
- $T \cup U$
- $U \setminus E$
- $\bar{E}$

Concept check: answers

- $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$, the even natural numbers
- $T = \{n \in \mathbb{N} \mid n < 10\}$
- $U = \{1, 2, 3\}$

What are the following sets?

- $E \cap T$
- $T \cup U$
- $U \setminus E$
- $\bar{E}$

- $\{0, 2, 4, 6, 8\}$
- $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\{1, 3\}$
- $\{n \in \mathbb{N} \mid n \text{ is odd}\}$