First-Order Logic

Predicates and Quantifiers

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Overview

1 DNF and CNF
- Propositions in Normal Form (3.4.1)

2 Extending our language

3 Translating to FOL

4 Proof rules for quantifiers

Refresher: validity and satisfiability

A propositional formula is *valid* if it is true under every possible assignment of truth values to its atoms. (All rows in the truth table come out T.)

A propositional formula is *satisfiable* if it is true under at least one truth assignment. (Some row in the truth table comes out T.)

Checking validity and satisfiability: a hard problem!

Disjunctive normal form

Definition: A formula in *disjunctive normal form* is an OR of terms, where each term is an AND of variables or negations of variables.

$(A \land B \land \neg C) \lor (\neg B \land C)$

$A \lor B \lor (A \land B \land \neg C)$

Not in DNF: $(A \land B) \lor \neg(B \land C)$

Disjunctive normal form is universal

Theorem: For every formula, there is an equivalent formula written in DNF.

Proof: You can read the terms off of the truth table, turning each "true" row into a conjunction of literals.

A	B	C	value		
F	**F**	**F**	**F**		
F	**F**	**T**	**T**	$\leftarrow$	$\neg A \wedge \neg B \wedge C$
F	**T**	**F**	**F**		
F	**T**	**T**	**F**		
T	**F**	**F**	**F**		
T	**F**	**T**	**T**	$\leftarrow$	$A \wedge \neg B \wedge C$
T	**T**	**F**	**T**	$\leftarrow$	$A \wedge B \wedge \neg C$
T	**T**	**T**	**F**		

$(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)$

Properties of disjunctive normal form

How big could the disjunctive normal form get? Big!

Definition: If every variable appears exactly once in every term in a disjunctive normal form expression, then it is in *full disjunctive normal form*.

Book	Wikipedia/me
disjunctive form	disjunctive normal form
disjunctive normal form	full disjunctive normal form

Given a formula in DNF (disjunctive normal form), can we determine whether it is satisfiable? Valid? Satisfiability is easy—a single term tells us a satisfying assignment. Validity is not obvious—a given term might exclude an assignment, but perhaps another picks it up?

Conjunctive normal form

Definition: A formula in *conjunctive normal form* is an AND of clauses, where each clause is an OR of variables or negations of variables.

$(\neg A \vee \neg B \vee C) \wedge (B \vee \neg C)$

$\neg A \wedge B \wedge (\neg A \vee C)$

Not an example: $\neg A \vee B \wedge (\neg A \vee C)$

Conjunctive normal form is universal

Theorem: For every formula, there is an equivalent formula written in CNF.

Proof: Negate the truth table. Write in DNF. Negate formula via DeMorgan's law. QED.

A	B	C	value	negated		
F	**F**	**F**	**T**	**F**		
F	**F**	**T**	**F**	**T**	$\leftarrow$	$\neg A \wedge \neg B \wedge C$
F	**T**	**F**	**T**	**F**		
F	**T**	**T**	**T**	**F**		
T	**F**	**F**	**T**	**F**		
T	**F**	**T**	**F**	**T**	$\leftarrow$	$A \wedge \neg B \wedge C$
T	**T**	**F**	**F**	**T**	$\leftarrow$	$A \wedge B \wedge \neg C$
T	**T**	**T**	**T**	**F**		

DNF for negated: $(\neg A \wedge \neg B \wedge C) \vee \ldots$

CNF: $(A \vee B \vee \neg C) \wedge \ldots$

Properties of conjunctive normal form

How big could the conjunctive normal form get? Big.

Definition: If every variable appears in every clause in a conjunctive normal form expression, then it is in *full conjunctive normal form*.

Book	Wikipedia/me
conjunctive form	conjunctive normal form
conjunctive normal form	full conjunctive normal form

Given a formula in CNF (conjunctive normal form), can we determine whether it is satisfiable? Valid? Validity is easy now—a single clause throws out an assignment, so a single clause makes the formula not valid. Satisfiability is not so clear—each clause knocks out some assignments, but not clear if the set of clauses miss anything.

First-order Logic

The language of *propositional* logic: atoms and connectives. Every formula is either an atom, or one or more formulas related by a connective. $p \wedge q \rightarrow r$

The language of *first-order* (or *predicate*) logic:

- Variables: $x, y, n, \ldots$
- Function symbols: $f(x)$, $plus(a, b)$, ... (sometimes with notation)
- Predicate symbols: $P(x)$, $R(x, y)$, $\mathrm{Prime}(n)$, ... propositions with placeholders
- Quantifiers: $\forall$, $\exists$
- ... and the same old connectives as before

Technical specification

A well-formed *term* in first-order logic is

- a variable (x, y, n, ...), or
- a function symbol applied to the correct number of terms ($f(x)$, $plus(x, y)$, ...), or
- a constant symbol (0, 1, $\emptyset$, ...)

Terms represent "things."

A well-formed *formula* in first-order logic is

- a predicate symbol applied to the correct number of terms ($R(x, y)$, $Prime(n)$, ...), or
- one or more formulas joined by a connective ($P(x) \wedge Q(y)$, $\neg R(x, y)$, ...), or
- a quantifier, followed by a variable, followed by a formula ($\forall x : \mathbb{N}, P(x) \wedge Q(x)$)

Formulas represent "statements." (Like propositions?)

Concept Check

Let $=$ and R be predicate symbols and $+$ and f be function symbols.

Which of the following are well-formed formulas?

- $x = 0 \vee x = 1 \vee x = 2$
- $f(x) \wedge f(y)$
- $\forall x : \mathbb{Z}, x + 0$
- $\exists x : \mathbb{Z}, \forall y : \mathbb{Z}, R(f(x), f(y))$
- $\forall x \wedge y = 2$

Translations

From day 1:

- There is a perfect square whose final digit is 4.
  $\exists x : \mathbb{N}, PS(x) \land (fd(x) = 4)$
- Every number is either prime or the product of two other numbers.
  $\forall n : \mathbb{N}, Prime(n) \lor \exists p\, q : \mathbb{N}, n = p \cdot q$
- Every number is either prime or the product of two *smaller* numbers.
  $\forall n : \mathbb{N}, Prime(n) \lor \exists p\, q : \mathbb{N}, (p < n) \land (q < n) \land (n = p \cdot q)$
- Every even integer greater than two is the sum of two primes.
  $\forall n : \mathbb{N}, Even(n) \land (n > 2) \to \exists p\, q : \mathbb{N}, Prime(p) \land Prime(q) \land (n = p + q)$

Try a few yourself!

You can make up some predicate and function symbols, like $TD(n)$ for "has two digits".

- $313(x^3 + y^3) = z^3$ has no solution when $x, y, z \in \mathbb{Z}^+$.
- There is a two-digit perfect square whose final digit is 4.
- Every prime number greater than 2 is odd.

Try a few yourself!

- $313(x^3 + y^3) = z^3$ has no solution when $x, y, z \in \mathbb{Z}^+$.
  $\neg\exists x\, y\, z : \mathbb{Z}^+, 313(x^3 + y^3) = z^3$
- There is a two-digit perfect square whose final digit is 4.
  $\exists n : \mathbb{N}, TD(n) \wedge PS(n) \wedge (fd(n) = 4)$
- Every prime number greater than 2 is odd.
  $\forall n : \mathbb{N}, Prime(n) \wedge (n > 2) \rightarrow Odd(n)$

forall proof rules

Introduction: To **prove** a forall goal $\forall x : T, G(x)$:
Suppose you have a (new, freshly named) $x : T$ in your context, and prove $G(x)$ for that new x.

I want to show that every number is either prime or the product of two other numbers. Suppose n is a number. Show that n is prime or n is the product of two other numbers.

Elimination: To **use** a forall hypothesis $\forall x : T, H(x)$:
If $t : T$ is any term of the right type, then you can add a hypothesis $H(t)$.

I know that every number is either prime or the product of two other numbers. Therefore, I know that either 2 prime or 2 is the product of two other numbers. I know that either 5 is prime or 5 is the product of two other numbers...

Exists proof rules

To **prove** an existential goal $\exists x : T, G(x)$:
Provide a witness.

I want to show that there is a perfect square whose final digit is 4. I claim that my witness is 64. Then, I must show that 64 is a perfect square and the final digit of 64 is 4.

Elimination: To **use** an existential hypothesis $\exists x : T, H(x)$:
you can create a (new, freshly named) $t : T$, and add a hypothesis $H(t)$. "Give a name to the witness."

I know that there is a perfect square whose final digit is 4. Let's call this perfect square *ps*. I know that *ps* is a perfect square and the final digit of *ps* is 4.