Proofs in Propositional Logic

Robert Y. Lewis

CS 0220 2024

January 31, 2024

Overview

Proof vs Truth

For today, we're going to think about propositional formulas, like $p \wedge q \rightarrow r$.

We saw a way to evaluate when these kinds of formulas are *true*: truth tables.

Today, we'll see a way to *prove* these kinds of formulas. Why the distinction?

Goals and Hypotheses

Suppose we have a proposition G that we want to prove.

The structure of G determines what we need to do to prove it.

Suppose we know a proposition H.

The structure of H determines what we can do with this fact.

During a proof, we might have multiple things that we want to prove (*goals*). Associated to each goal, there is a list of things we know (a list of *hypotheses*, making up a *context*).

The Proof Game

Start: one goal, zero hypotheses.

Aim: all goals completed.

Moves: *proof rules*, to change *proof state*.

An example from last week: "there is a perfect square whose final digit is 4." Proof rule: to prove an existential, provide a witness: 8^2. Goal becomes, "the final digit of 8^2 is 4." (True by computation.)

Propositional Logic

Let's make this more precise.

We introduced the language of propositional logic: formulas built out of atoms and connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

What are the proof rules associated with these symbols?
Two categories of rules. *Introduction rules* say how to prove a goal of a certain form.
Elimination rules say how to use a hypothesis of a certain form.

"And" Introduction

If your goal is to prove $P \wedge Q$: first prove P, then prove Q. (Turns one goal into two smaller goals.)

"Or" Introduction

If your goal is to prove $P \vee Q$, there are two rules you can follow:

- Prove P. ("left")
- Prove Q. ("right")

Both rules turn one goal into one smaller goal.

An example: prove $(1 + 1 = 2 \vee 1 + 1 = 3) \wedge (2 \cdot 2 = 5 \vee 2 \cdot 2 = 4)$.

Implication Introduction

To prove $P \to Q$: *assume* P (a new hypothesis), and show Q (a new goal).

Goal: if x is even, then x^2 is even. Suppose x is even. We use this fact to show that x^2 must be even, because ....

To show $P \leftrightarrow Q$: show $P \to Q$ and $Q \to P$ (two goals).

Atoms

If you have a hypothesis P in your context, you can close a goal of P. ("By assumption")

Goal: if x is even, then x is even. Suppose x is even. Our goal is now to show that x is even. This follows by assumption.

In Lean

Introduction rules in Lean:

- and intro: `split_goal`
- or intro: `left`, `right`
- implication intro: `assume h`
- iff intro: `split_goal`

"And" Elimination

If you know $P \wedge Q$, you know two things:

- P
- Q

Yes, this sounds silly to say out loud. We usually don’t think about this.

In terms of proof state: turns one hypothesis into two smaller hypotheses.

"Or" Elimination

This one's more interesting!

If you know $P \vee Q$, and your goal is G, you can *reason by cases*. That is: if you show $P \rightarrow G$, and you show $Q \rightarrow G$, then you have shown G.

In terms of proof state: creates two goals, each with a new hypothesis.

Implication Elimination: modus ponens

If x is prime, then $x \geq 2$. x is prime. Therefore, $x \geq 2$.

General pattern: if you know $P \to Q$ and you know P, then you know Q.

Adds a hypothesis.

Alternate phrasing: if your goal is to show Q, and you know $P \to Q$, it suffices to show P.

Changes the goal.

(Iff elimination is easy: if you know $P \leftrightarrow Q$, then you know $P \to Q$ and $Q \to P$.)

Getting comfortable with contradiction

We live in a world where things make sense. (...)

In our sensible world, some statements are true and some are false. But none are true *and* false.

So if we can prove both a proposition and its negation, we're living in nonsense land. Anything follows.

Negation elimination and introduction

Elimination proof rule: if you know P and you know $\neg P$, you can prove anything (i.e. close any goal).

Introduction proof rule: if your goal is to prove $\neg P$, you can assume P, and show "false". Proof by contradiction!

Example proof by contradiction

Proposition: $\sqrt{2}$ is not rational.

We prove that $\sqrt{2}$ is not rational by contradiction. Suppose $\sqrt{2}$ is rational. By the definition of "rational", that means $\sqrt{2} = p/q$ where p and q are integers. Furthermore, we can choose p and q to be in lowest terms so they have no factors in common. Squaring both sides, we get $2 = p^2/q^2$ or $2q^2 = p^2$. Since q^2 is an integer, and p^2 is an integer times 2, p^2 is even. By a similar argument to the one for odd squares (from a few lectures ago), that means p must be even. If p is even, p^2 must be divisible by 4. Since $2q^2$ is divisible by 4, q^2 must be divisible by 2 (the other factor of two must be there). That means both p and q are even. But, then p/q is not in lowest terms. Since we already asserted that p/q is in lowest terms when p and q were chosen, we've reached a contradiction. Therefore, $\sqrt{2}$ must be irrational.