Propositional Logic

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Overview

What is propositional logic?

Today, we care about the *shape* of propositions, not their content.
We have:

- *Atoms*: propositions that we can't decompose any further
- *Connectives*: operators that take in propositions and produce new propositions

We build compound propositions out of smaller ones. "It is snowing, and I am cold."

"Boolean logic": every proposition is either true or false. The truth of a compound proposition depends on the truth of its components.

Notation for propositional logic

Our connectives:

- **NOT:** $\neg P$
- **AND:** $P \wedge Q$
- **OR:** $P \vee Q$
- **IMPLIES (IF-THEN):** $P \rightarrow Q$
- **IFF:** $P \leftrightarrow Q$

$P \wedge \neg Q$
$P \wedge (\neg Q)$
$(A \wedge B) \rightarrow (C \vee D)$

Truth tables

Truth tables help convey how to interpret logical statements.

P	$\neg P$
F	**T**
T	**F**

P	Q	$P \wedge Q$
F	**F**	**F**
F	**T**	**F**
T	**F**	**F**
T	**T**	**T**

Like a multiplication table, but flattened (so easier to apply to more inputs.)

OR

P	Q	$P \vee Q$
F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**T**

In English, often rendered as “P or Q or both”. Inclusive or.

P	Q	P XOR Q
F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**F**

Exclusive or, too.

Implication

Perhaps counterintuitive.

X	Y	$X \to Y$
F	**F**	**T**
F	**T**	**T**
T	**F**	**F**
T	**T**	**T**

This is known as the *material conditional.* Compare:

If it is raining outside, then I have my umbrella. (material)

If it were raining outside, then I would have my umbrella. (counterfactual)

Math deals with material conditionals, so this class does too.

Implication

X	Y	$X \to Y$
F	**F**	**T**
F	**T**	**T**
T	**F**	**F**
T	**T**	**T**

Claim: If p is a prime number greater than 2, then p is odd.

"Should" be true for every number p.

Definition: In "$X \to Y$", X is called the *hypothesis* and Y the *conclusion*.

Can you think of an equivalent way to write this using $\wedge$, $\vee$, $\neg$?

Implication

X	Y	$X \to Y$
F	**F**	**T**
F	**T**	**T**
T	**F**	**F**
T	**T**	**T**

X	Y	$\neg X$	$\neg X \vee Y$
F	**F**	**T**	**T**
F	**T**	**T**	**T**
T	**F**	**F**	**F**
T	**T**	**F**	**T**

Truth table for iff

X	Y	$X \leftrightarrow Y$
F	**F**	**T**
F	**T**	**F**
T	**F**	**F**
T	**T**	**T**

Example: A number is divisible by three *if and only if* the sum of its digits is divisible by three.

Definitions have this form.

Truth tables for compound formulas

To build a truth table for a compound formula:

- Find the atoms, and make those columns.
- Find the compound subformulals, and make those columns.
- Fill in all combinations of truth values for the atoms.
- Propogate right, column by column.

An example in Lean

Natural language translations

Translating English sentences into propositional logic seems straightforward: choose your atomic propositions. Then "and" becomes $\wedge$, "or" becomes $\vee$, ...

But it's not always quite so easy. Define:

- T := "Rob likes t rexes"
- S := "Rob likes stegosauruses"
- F := "Rob's dinosaur zoo is full"
- W := "Rob wants more dinosaurs"

"Rob likes t rexes and stegosauruses." $T \wedge S$

"Rob wants more dinosaurs, but his zoo is full." $W \wedge F$

"If Rob does not want more plants, then he doesn't like t rexes or stegosauruses."
$\neg W \rightarrow \neg(T \vee S)$ or maybe $\neg W \rightarrow (\neg T \wedge \neg S)$