Probability and Independence

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Overview

1. Independence (17.7)
2. Alternative Formulation (17.6.1)
3. Mutual Independence (17.6.3)
4. A philosophical digression
Probability refresher

A sample space is a set of possible outcomes. An event is a subset of the sample space. A probability function assigns each outcome a probability between 0 and 1.
Conditional probability

**Definition:** The conditional probability of event $A$ given event $B$ (with $\Pr[B] > 0$) is:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$ 

Conceptually, if we limit ourselves to the outcomes in $B$, how likely is an outcome in $A$?

Example:

- **$A$:** Die shows a number divisible by 3. $\Pr[A] = 1/3$. (3 and 6 from the six possibilities.)
- **$B$:** Die shows an odd number. $\Pr[B] = 1/2$.
- What does $\Pr[A \cap B]$ mean? Die shows an odd number divisible by 3. $\Pr[A \cap B] = 1/6$ (only 3).
- What does $\Pr[A|B]$ mean? Die shows a number divisible by 3 given that it’s odd. $\Pr[A|B] = 1/3$ (probability of picking 3 from 1, 3, 5). Also, $\frac{1/6}{1/2} = \frac{1}{6} \times 2 = \frac{1}{3}$. 

Independence

**Definition:** Event $A$ is *independent of* event $B$ iff

$$Pr[A|B] = Pr[A].$$

If $Pr[B] = 0$, we say it is independent of any other event including itself.

Example:

- **A:** Die shows the maximum or minimum number. $Pr[A] = 1/3$. (1, 6 from the six possibilities.)
- **B:** Die shows an odd number. $Pr[B] = 1/2$.
- $Pr[A|B] = 1/3$. (1 from 1,3,5.) So, $A$ and $B$ are independent.
- **C:** Die shows an even number. $Pr[C] = 1/2$.
Independent events multiply

**Theorem:** A is independent of B iff

\[ \Pr[A \cap B] = \Pr[A] \cdot \Pr[B]. \]

Proof: By cases.

- **Case 1:** If \( \Pr[A] = 0 \) or \( \Pr[B] = 0 \), then \( \Pr[A \cap B] = 0 \). Equality and independence are both achieved.

- **Case 2:** Otherwise,

  \[ A \text{ is independent of } B \iff \Pr[A|B] = \Pr[A] \text{ (def. independence)} \]
  \[ \iff \Pr[A \cap B]/\Pr[B] = \Pr[A] \text{ (def. cond. prob.)} \]
  \[ \iff \Pr[A \cap B] = \Pr[A] \cdot \Pr[B] \text{ (multiplying across)} \]
Independent coin flips

I flip a fair coin twice.

- $A$: first flip is heads.
- $B$: second flip is heads.

Independent events?
Hash collisions

I have a perfect hash function and two pieces of data $a$ and $b$ to insert into a hashtable.

- $A$: $a$ has a hash collision.
- $B$: $b$ has a hash collision.

Independent events?

Independence: “knowledge about one event does not give us knowledge about another.”
Mutual Independence

**Definition:** A set of events $E_1, E_2, \ldots, E_n$ is *mutually independent* iff for all subsets $S \subseteq [1, n],$

$$\Pr \left[ \bigcap_{j \in S} E_j \right] = \prod_{j \in S} \Pr[E_j].$$

Example: If we toss $n$ fair coins, the tosses are mutually independent iff for every subset of $m$ coins, the probability that every coin in the subset comes up heads is $2^{-m}$. 
Pairwise independence isn’t mutual independence

If $A$ is independent of $B$ and $C$, and $B$ and $C$ are independent of each other, how could $A$, $B$, and $C$ not be independent??

Example: Jania, Tyler, Carmen each pick a bit 0/1 uniformly at random.

- $A$: Jania + Tyler $\equiv 1 \pmod{2}$
- $B$: Tyler + Carmen $\equiv 1 \pmod{2}$
- $C$: Jania + Carmen $\equiv 1 \pmod{2}$

Claim 1: These events are all pairwise independent.

For example, $\Pr[A] = 1/2$. $\Pr[A|B] = \frac{1}{4}/\frac{1}{2} = 1/2$.

Claim 2: These events are not mutually independent.

$\Pr[A \cap B \cap C] = 0$. Not 1/8!

$k$-wise does not imply $(k + 1)$-wise mutual independence.
What are we even measuring?

What does it mean to say:

- The probability that a sequence of 5 coin flips will be all heads is \(\frac{1}{32}\)?
- The probability that it will rain tomorrow is .8?
- The probability that \(2^{6972607} - 1\) is a prime number is...?

Two interpretations: *frequentist* vs *Bayesian*. 
Frequentism vs Bayesianism

The frequentist: probability statements only apply to *repeatable* events. Coin flip example makes sense: if we repeatedly sample from the sample space, the “all heads” event will happen in about $1/32$ of the samples. Rain question is meaningless. Prime number question is either 0 or 1.

The Bayesian: probability statements describe a subjective *belief*/level of confidence. I’d take either side of a bet with 1:32 odds on the coin flip sequence. I think it’s four times more likely to rain tomorrow than not. There’s an objective answer to the prime number question, but based on my current information/computation ability, ...

Mathematical probability: tries to be agnostic between these two interpretations.