# Counting Trees

Robert Y. Lewis

CS 0220 2024

April 3, 2024

# Overview

# Format

Different format today. We’re going to look at one problem and some failed attempts to solve it, along with a solution that actually works.

# Reminder: graphs and trees

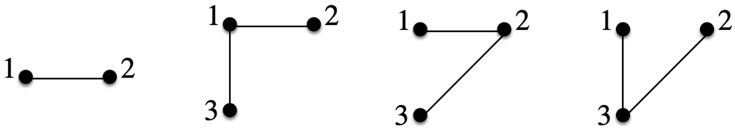
A *graph* $G$ has a set of *vertices* $V(G)$ and an adjacency relation on those vertices (equivalently, a set of edges).

A *tree* is a *connected* graph with no *cycles*.

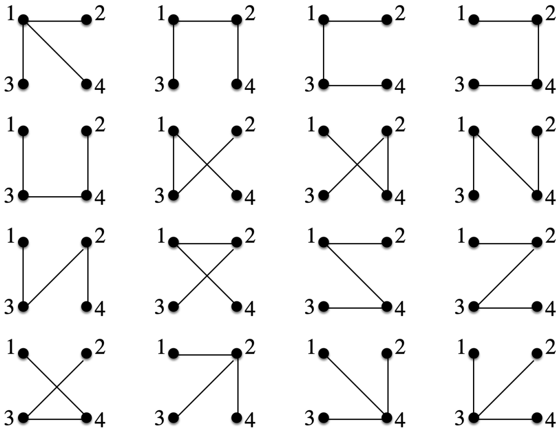
# Counting trees

How many ways can we connect *n* vertices together into a tree?

Trees on 2 and trees on 3:



# Trees on 4



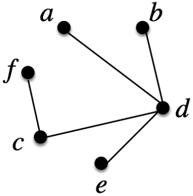
# What we know so far

Trees: Connected, acyclic.

| $n$ | trees |
|---|---|
| 2 | 1 |
| 3 | 3 |
| 4 | 16 |

# Depth and parents

We can define any vertex of a tree to be its root.




**Definition**: Given a tree $G$ and a choice of root $r \in V(G)$, the *depth* of $u \in V(G)$, $\mathrm{dep}_r(u)$ is the length of the simple path from $r$ to $u$.

Depth is well defined because every pair of nodes in a tree has a unique simple path between them.

**Definition**: Given a tree $G$ and a choice of root $r \in V(G)$, $u$ is the *parent* of $v$ if $(u, v) \in E(G)$ and $\mathrm{dep}_r(u) = \mathrm{dep}_r(v) - 1$.

# Choosing edges

Ok, first thought. A tree on $n$ vertices has $n-1$ edges out of all possible edges, since each vertex (except the root) is connected to exactly one parent. So pick these edges:

$$\binom{\binom{n}{2}}{n-1}.$$

| $n$ | trees | |
|---|---|---|
| 2 | 1 | |
| 3 | 3 | |
| 4 | 20 | $> 16$ |

We counted cycles that aren't trees.

# Ways to add a vertex

So, let's be careful to only generate trees. Here's the thought. Consider a tree on $n$ vertices. We can add vertex $n+1$ and connect it into the tree $n$ different ways. We're guaranteed that the new graph is connected and acyclic (a tree!).

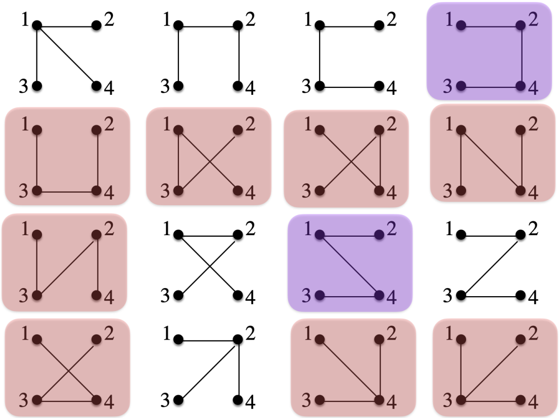
So, 2 vertices make 1 tree. Adding the 3rd vertex creates 2 times more. Adding the 4th vertex creates 3 times more.

Generalizing, we get $(n-1)!$ trees.

| $n$ | trees | |
|---|---|---|
| 2 | 1 | |
| 3 | 2 | $< 3$ |
| 4 | 6 | $< 16$ |

Now, we're only counting trees, but we're missing some trees. In particular, we're missing trees where the last vertex is somewhere in the middle.

# Generated 4 trees



# More careful growing

When we add in vertex $n + 1$, the other $n$ vertices might not be a tree. They might be a forest. In general, vertex $n + 1$ will have one edge to each connected component in the forest. For the edge to connected component $i$, it will have a choice of which of the vertices in the connected component to connect to.

You can *almost* write down an expression, but it's complicated and not clear how to simplify it. The basic idea is consider all the ways of making a tree with $n'$ vertices for all $n' \leq n$, then all the ways $n$ vertices can be partitioned into clusters, then sum and multiply...

But even the question of how many partitions there are for $n$ items is hairy. See: https://oeis.org/A000110 .

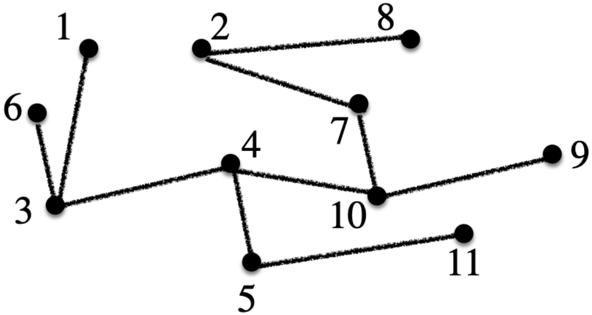
# Better way to look at it

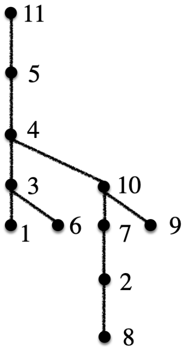
Sometimes there’s just a better way to look at it. It’s definitely not obvious (to me!). But it’s clever and gives a nice clean answer.

1. Create a “normal form” for trees. That way, we can at least notice when two different trees are actually the same.
2. Find a compact encoding for these trees. Here, “compact” means no extraneous information.
3. Then, we can show we have a bijection (!) between trees and the encoding.
4. If we’re lucky, it’ll be easier to count encodings than trees.

# Normal form for trees

Choose vertex *n* to be the root. Order children smallest (left) to biggest (right). There are no other choices.



# Encoding a tree

Every vertex has a unique parent. So, we could just give the parent for each vertex: 3 7 4 5 11 3 10 2 10 4.

List $n - 1$ numbers (since root has no parent), each with a choice of $n - 1$ numbers (can't pick yourself!). That's $(n - 1)^{n-1}$.

Anything extraneous? Yes, can encode a loop: 3 1 2 5.

# Sanity check

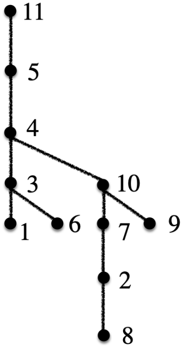
Could it be $(n-1)^{n-1}$?

| $n$ | trees | |
|---|---|---|
| 2 | 1 | |
| 3 | 4 | $> 3$ |
| 4 | 27 | $> 16$ |

Yeah, we're definitely overcounting. But we're guaranteed not to undercount. Every tree *has* a representation in this scheme. But some representations do not produce trees. It's not a bijection.

# Prufer rule

Repeat the following procedure $n - 2$ times. Find the smallest valued leaf. Write down its parent. Delete the leaf.

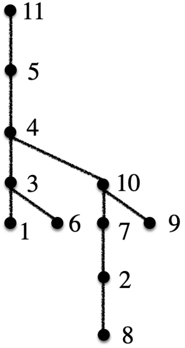



3 3 4 2 7 10 10 4 5

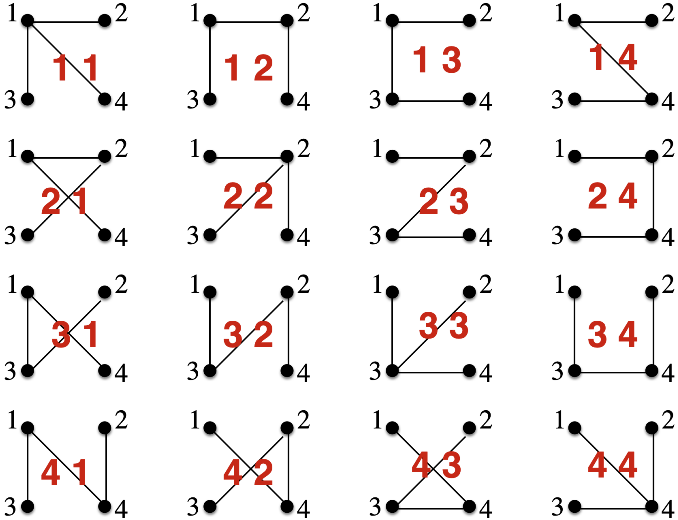
# Recover a tree

We can process any tree into a list. But can we recover the tree from the list? Before we prove that we can, let's do an example:

3 3 4 2 7 10 10 4 5

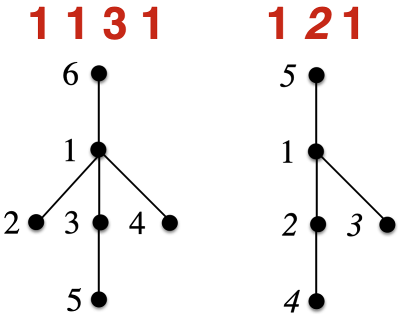


# 4 by 4



# Thinking inductively

Here's a 6-vertex tree in Prufer encoding: 1 1 3 1.




In what sense is it built out of a 5-vertex encoding? Take the vertex $x$ that is the "first" leaf. Here, $x = 2$. Remove it, then renumber the vertices, decrementing anything larger than $x$. The thing to note is that the resulting tree and encoding still match!

# Proof

**Theorem**: For every string $a \in [1, n]^{n-2}$ ($n \geq 2$), there is a unique tree $T$.

**Proof**: Build-down induction on $n$.

Base Case ($n = 2$): There is only one tree (a 1–2 segment) and only one encoding string (the null string).

Inductive Step ($n + 1$): Consider a string $a$ of length $n - 1$. In the tree $T$ encoded by $a$, the leaf with the smallest label $x$ must be linked to $a_1$.

Consider the string $a'$ formed by removing $a_1$ from $a$ and then subtracting one from every value in $a$ that's larger than $x$. By the inductive hypothesis, there is a unique tree $T_0$ constructed from $a'$. We can construct a unique tree $T$ from $T_0$ by adding 1 to the values in $T_0$ that are $x$ or above, then adding the edge $(x, a_1)$.

# Summing up

We have a scheme for encoding trees as lists. It always works. We have a scheme for turning lists into trees. It always works. We have a bijection.

How many lists? $n-2$ choices of numbers from 1 to $n$. So, $n^{n-2}$. Does that fit?

| $n$ | trees |
|---|---|
| 2 | 1 |
| 3 | 3 |
| 4 | 16 |