Pigeonhole Principle

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Overview

What's coming up

- We've just started talking about *combinatorics*, the art of counting things.
  - Our move: get good at counting certain kinds of things, like sequences. Then count other things by finding bijections to sets of sequences.
- $\binom{n}{k}$: the number of ways to pick k items from a set of n options.
- Leads into *probability*.
  - In the finite case, can be thought of as "weighted counting."
- A general framework (probability spaces, random variables), on which we'll define many operations in ways that generalize.

Puzzle

A drawer in a dark room contains red socks, green socks, and blue socks. You are going to grab k socks, then go someplace light. How many socks must you take with you to be sure that you have at least one matching pair?

Two might be enough. But it’s not guaranteed. After all, one might be one red, one green. Three is not enough. Might grab one of each. At four, one of the colors *must* repeat—you have a matching pair!

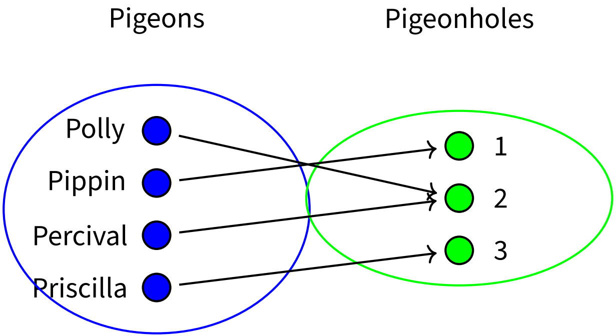
Pigeonhole principle

If there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole.

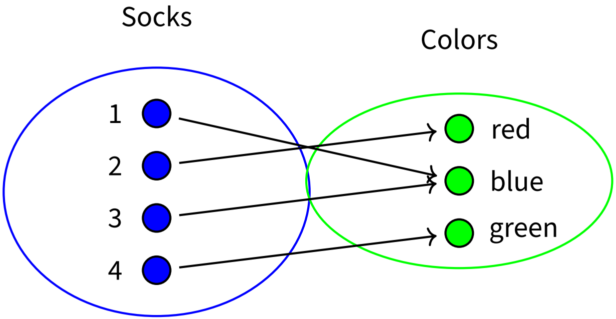
Why pigeons? A linguistic artifact. «Principe des tiroirs », “ladenprincipe”: think “drawers” or “cabinets.”

We will use it as a tool for reasoning about the relationships between the sizes of sets.

Pigeonhole principle as a relation



Sock example



Pigeonhole principle

Rule: If $|A| > |B|$, then, for every function $f : A \to B$, there exist two different elements of A that are mapped by f to the same element of B.

Recall: If a function from A to B is injective (no collisions), then $|A| \leq |B|$.

By the contrapositive of this statement, if $|A| > |B|$ (it's not the case that $|A| \leq |B|$), then a function from A to B has collisions (is not injective).

Australian sheep

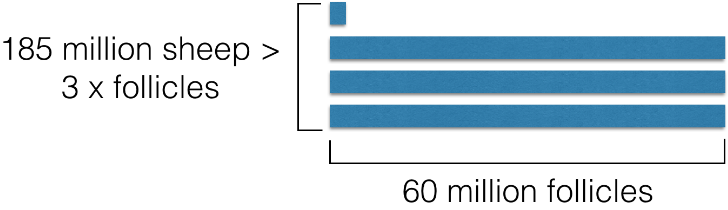
There’s a pair of sheep in Australia that have precisely the same number of hairs.

Sheep have up to 60 million wool follicles. Australia has 70 million sheep. Pigeonhole principle says there must be (at least!) two that match up.

All the sheep in China

China has 185 million sheep, each with up to 60 million follicles. Same argument as Australia applies. But, we can go further.

There must be a *foursome* of sheep with precisely the same number of hairs on their body. Argument: Even if we spread things out maximally, there's some follicle count with 4 sheep in it.



Generalized Pigeonhole Principle

Rule: If $|A| > k|B|$, then every function $f : A \to B$ maps at least $k + 1$ different elements of A to the same element of B.

Lossless compression

An example from CS!

Let B be the set of bit sequences of length at most n. A *compression algorithm* is a function $f : B \to B$ that intends to map “interesting” sequences to shorter sequences. A *lossless* compression algorithm is one that is invertible.

By the pigeonhole principle: every lossless compression algorithm that compresses at least one input must expand some other input.

Why isn’t this a problem?

Hash functions

Another classic CS application: hash maps.

Idea: store elements of A in an array of length n. Compute the index of element a by $h(a)$, where $h : A \to \mathbb{N}$ is a *hash function*.

Pigeonhole principle tells you when you're guaranteed to have a collision: there exist two distinct elements with the same hash value exactly when $|A| > n$.

But this isn't the analysis we really want to do. How likely is a collision? How big should n be, in general?

25 2-digit numbers

68	34	41	89	99
0	95	56	90	43
41	37	94	94	69
15	96	85	76	30
27	24	75	63	1

Two pairs with the same sum?

$$76 + 30 = 106 = 43 + 63$$
$$37 + 94 = 131 = 41 + 90$$

Lots of others.

Analysis

List of 25 2-digit numbers. Must two different pairs have the same sum?

A: How many possible pairs? $\binom{25}{2} = 25 \times 24/2 = 300$. In general, n numbers can be paired in $\binom{n}{2}$ ways.

B: How many possible pairwise sums? Smallest is $0 + 0 = 0$, largest is $99 + 99 = 198$. So, 199 possible values. In general, pairs of d-digit numbers can sum to $2 \times 10^d - 1$ possible values.

Since $300 > 199$, by the pigeonhole principle, some possible sum *must* be the result of two different pairs.

40 10-digit numbers

372995585	2234938293	7149708291	7060913492
2931952606	3111391181	951202341	735405394
9217967649	824907338	5014746657	1237286631
522671716	8407467172	8707242896	683510550
4879527699	5950653401	5161166931	816710053
1132327190	5154516759	2790970967	2560087807
8425382298	4088307493	3641040020	2603650330
3153589691	8144289385	4677056137	4058664392
9202688143	144564604	6259750302	2210904697
6037630341	498877221	186119657	71015872

Two *subsets* with the same sum?

Analysis

List of 40 10-digit numbers. Must two different subsets have the same sum?

A: How many possible subsets? $2^{40} = 1.1e + 12$. In general, *n* numbers can produce 2^n subets.

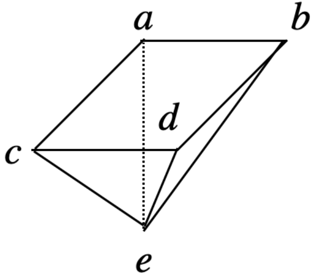
B: How many possible subset sums? Smallest is $0 + 0 = 0$, largest is $40 \times 9999999999 = 4.0e + 11$. In general, up to *n* *d*-digit numbers can sum to $n \times 10^d$ possible values.

Since $1.1e + 12 > 4.0e + 11$, by the pigeonhole principle, some possible sum *must* be the result of two different subsets.

Graphs and polyhedra

Definition: A *graph* G consists of a set of *vertices* $V(G)$ and *edges* $E(G)$.

This terminology comes from descriptions of flat-sided solid shapes—polyhedra.



$\{a, b, c, d, e\}$ are the vertices, $\{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}\}$ are the edges.

Adjacency relation

$E(G) \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}$.

u adjacent to v in G iff $\{u, v\} \in E(G)$.

Is the "adjacent to" relation reflexive? No. Symmetric? Yes! Transitive? Depends. Empty is graph is! Otherwise, no, because symmetry and non-reflexive.

A *simple path* between vertices v_0 and v_k is a sequence of vertices $(v_0, v_1, \dots, v_k) \in V(G)^{k+1}$ such that

- $v_0, \dots, v_k$ are all distinct;
- for each $i < k$, v_i and v_{i+1} are adjacent.

A graph is *connected* if there is a simple path between every pair of vertices.

Trees

A *simple cycle* is a simple path between vertices v_0 and v_k with the added constraint that v_0 and v_k are adjacent.

A *tree* is a connected graph that has no cycles.