Binomial Theorem, Inclusion/Exclusion

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Overview

Choice

$$\binom{n}{k}$$

The number of k-element subsets of an n-item set. "n choose k".

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomials to powers: Examples

$$\begin{aligned}
(a+b)^2 &= aa + ab + ba + bb \\
&= a^2 + 2ab + b^2
\end{aligned}$$

$$\begin{aligned}
(a+b)^3 &= aaa + aab + aba + abb \\
&\quad + baa + bab + bba + bbb \\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}$$

$$\begin{aligned}
(a+b)^4 &= aaaa + aaab + aaba + aabb \\
&\quad + abaa + abab + abba + abbb \\
&\quad + baaa + baab + baba + babb \\
&\quad + bbaa + bbab + bbba + bbbb \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

How about $(a+b)^n$? How many terms consist of exactly k bs? Since it's all combinations of an a and b in each position, there are $\binom{n}{k}$ such terms.

Binomial theorem

Theorem: For all $n \in \mathbb{N}, a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Sometimes $\binom{n}{k}$ called the *binomial coefficient* because of this connection.

Pascal's Triangle

$n = 0$							1						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				1		3		3		1			
$n = 4$			1		4		6		4		1		
$n = 5$		1		5		10		10		5		1	
$n = 6$	1		6		15		20		15		6		1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pets and sets

S: Set of all students in CS0220.

$D \subseteq S$: Set of all students in CS0220 who have a pet dog.

$C \subseteq S$: Set of all students in CS0220 who have a pet cat.

$D \cup C$: Set of all students in CS0220 who have a pet dog *or* cat.

$|D \cup C| = |D| + |C|$? Handles people who have neither correctly. Handles people who have one kind of pet correctly. Messes up on people who have both.

Formulas for union

What's wrong with each formula for $|C \cup D|$?

- $|C| + |D|$? Double counted people who have both.
- $|C \setminus D| + |D \setminus C|$? Skipped people who have both.
- $|C \setminus D| + |D \setminus C| + |C \cap D|$? Actually, that should work. But, set difference can be tricky.
- $|C| + |D| - |C \cap D|$? Nailed it. Correct for double counting

Inclusion-Exclusion rule for two sets

Rule: For two sets S_1 and S_2,

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|.$$

Example:

- $S_1 = \{\text{ Allie, Tyler }\}$: HTAs with an *l* in their name.
- $S_2 = \{\text{ Allie, Jania }\}$: HTAs with an *i* in their name.
- $S_1 \cap S_2 = \{\text{ Allie }\}$: HTAs with both an *i* and an *l* in their name.
- $S_1 \cup S_2 = \{\text{ Jania, Allie, Tyler }\}$: HTAs with either an *i* or an *l* in their name.
- $|\{\text{ Jania, Allie, Tyler }\}| = |\{\text{ Allie, Tyler }\}| + |\{\text{ Allie, Jania }\}| - |\{\text{ Allie }\}|$

Generalize to three sets

S: Set of all students in CS0220.

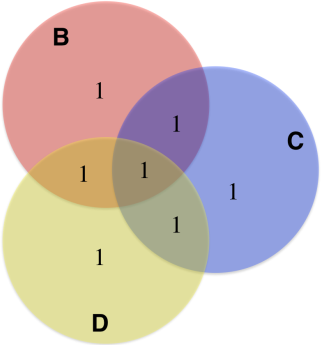
$D \subseteq S$: Set of all students in CS0220 who have a pet dog.

$C \subseteq S$: Set of all students in CS0220 who have a pet cat.

$B \subseteq S$: Set of all students in CS0220 who have a pet bunny.

How express $|B \cup C \cup D|$ in terms of size of *intersections* of sets?

Visual analysis



$$|B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D|$$

Inclusion-Exclusion rule for three sets

Rule: For three sets S_1, S_2, S_3,

$$\begin{aligned}|S_1 \cup S_2 \cup S_3| = \quad & |S_1| + |S_2| + |S_3| \\ & -|S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ & +|S_1 \cap S_2 \cap S_3|.\end{aligned}$$

Example:

- $S_1 = \{$ Jania, Allie, Carmen $\}$: HTAs with an *a* in their name.
- $S_2 = \{$ Allie, Joseph, Carmen, Tyler $\}$: HTAs with an *e* in their name.
- $S_3 = \{$ Jania, Allie $\}$: TAs with an *i* in their name.
- $S_1 \cap S_2 \cap S_3 = \{$ Allie $\}$: TAs with an *a* and an *e* and an *i* in their name.
- $|\{$ Jania, Allie, Joseph, Carmen, Tyler $\}| = |\{$ Jania, Allie, Carmen $\}| +$ $|\{$ Allie, Joseph, Carmen, Tyler $\}| + |\{$ Jania, Allie $\}| - |\{$ Allie, Carmen $\}|$ $-|\{$ Jania, Allie $\}| - |\{$ Allie $\}| + |\{$ Allie $\}|$

Sets of permutations

In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- $(4, 6, 5, 0, 1, 8, 3, 2, 9, 7)$ nope.
- $(0, 4, 6, 1, 8, 5, 9, 3, 7, 2)$ 04!
- $(3, 4, 2, 0, 5, 6, 1, 9, 8, 7)$ 42!
- $(3, 9, 4, 1, 2, 7, 0, 5, 6, 8)$ nope.
- $(0, 2, 6, 3, 7, 8, 4, 9, 5, 1)$ nope.

P_{60}: permutations of 0 through 9 that contain 60.

P_{04}: permutations of 0 through 9 that contain 04.

P_{42}: permutations of 0 through 9 that contain 42.

Want: $|P_{60} \cup P_{04} \cup P_{42}|$.

Inclusion-exclusion, constrained permutation

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60}| = ?$

Clever trick: In P_{60}, can view “60” as a unit. So, each element of P_{60} is a permutation of $\{1, 2, 3, 4, 5, 7, 8, 9, 60\}$. Therefore, $|P_{60}| = 9!$. $|P_{04}| = 9!$. $|P_{42}| = 9!$.

Pairwise intersections

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}| \\
&= 3 \times 9! \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60} \cap P_{04}| =$? Trick works again! Can view “604” as a unit. So, each element is a permutation of $\{1, 2, 3, 5, 7, 8, 9, 604\}$. Therefore, 8!.

$|P_{42} \cap P_{04}| =$? Trick works again! Can view “042” as a unit. So, 8!.

$|P_{60} \cap P_{42}| =$? Trick fails! Wait, no, just changes. Now, each element is a permutation of $\{1, 3, 5, 7, 8, 9, 60, 42\}$. Still 8!.

Three-way intersection

$$
\begin{aligned}
&|P_{60} \cup P_{04} \cup P_{42}| \\
&= |P_{60}| + |P_{04}| + |P_{42}| \\
&\quad -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}| \\
&= 3 \times 9! - 3 \times 8! \\
&\quad +|P_{60} \cap P_{04} \cap P_{42}|
\end{aligned}
$$

$|P_{60} \cap P_{04} \cap P_{42}| =?$. Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of $\{1, 3, 5, 7, 8, 9, 6042\}$. Therefore, $7!$.

$|P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720.$

n-way Inclusion-Exclusion

$|S_1 \cup S_2 \cup \cdots \cup S_n| =$

	the sum of the sizes of the individual sets
minus	the sizes of all two-way intersections
plus	the sizes of all three-way intersections
minus	the sizes of all four-way intersections
plus	the sizes of all five-way intersections, etc.

Hyper-mathy version:

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{X \in \mathcal{P}([1,n]) - \emptyset} (-1)^{|X|+1} \left|\bigcap_{i \in X} S_i\right|$$