Intro to Counting

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Overview

What have we done so far?

We started off talking about propositional (boolean) logic.

- Connected to circuits, foundations of computing
- Conditional statements: *everywhere* in CS
- Boolean satisfiability: a “perfect” generic problem

Then we talked about sets, relations, and proof techniques.

- A language to describe groups of objects and ways to manipulate them
- A sandbox for writing rigorous logical arguments
- Functions and relations, related terminology: essential for reasoning about program design

What have we done so far?

Then number theory: induction, modular arithmetic, encryption.

- A more interesting sandbox to talk about proofs, without losing ourselves in details
- On one level of abstraction, computers are number-crunching machines (functions)
- Computers like finite things; mathematicians like infinite things; integers are somewhere in between
- Canonical “hard and important” calculations related to encryption

Another connection between these topics: *discreteness*.

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	sitting in seats
$\mathbb{R}$	$\mathbb{Z}$
quantities	numbers
measuring	**counting**

General strategy for counting

- Get good at counting some categories of things.
- Use bijections to relate one of those to the problem at hand.
- Often: we'll count *sequences* (ordered pairs) $(a, b, c) \in A \times B \times C$.

Why??

- Analyze the running time or memory needs of our algorithms
- Make our algorithms more efficient: rule out unneeded checks

Product rule

Rule: If $P_1, P_2, \ldots, P_n$ are finite sets, then:

$$|P_1 \times P_2 \times \cdots P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|.$$

Example:

- beverage $=$ { coffee, tea, hot chocolate }
- size $=$ { small, medium, large, extra large }
- milk $=$ { yes, no }

If an order consists of a choice of beverage (one of 3), size (one of 4), and whether it has milk added (one of 2), how many different orders are there? $3 \times 4 \times 2 = 24$.

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Why are there 2^n binary strings of length n?

$$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}.$$

Product rule! $2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$.

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

- 3 different kinds of coffee.
- 9 different kinds of tea.

Total drink choices? $3 + 9 = 12$.
What if each can be small or large? $3 \cdot 2 + 9 \cdot 2 = 24$.

Counting Passwords

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: $\{$ a, b, ..., z $\}$. 26
- Letter symbol: letter $\times$ $\{$ upper, lower$\}$. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 =$ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 =$3e+12
- 8-symbol password: letter symbol $\times$ 7 letter/digit symbols. $52 \cdot 62^7 =$ 2e+14
- valid password: 6-symbol password $\cup$ 7-symbol password $\cup$ 8-symbol password. 5e+10 + 3e+12 + 2e+14 = 2e+14

Counting assignments

I have 4 different postcards and 5 different people. How many different ways can I give the postcards to the people?

There's a bijection between sequences and postcard assignments.

If the five people are Allie, Carmen, Jania, Joseph, and Tyler, I can give the first postcard to Allie, the second to Josh, the third to Ben, and the fourth to Josh. The sequence is: (Allie, Josh, Ben, Josh). Every assignment of the 4 postcards to the people becomes a sequence. Every sequence corresponds to a way to give the postcards to the people. Bijection. There are $5^4 = 625$ sequences (product rule), so there are 625 ways of assigning the postcards to people.

In general, n people, k distinct postcards, n^k assignments of the postcards to people.

Generalized product rule

If we add the constraint that no person gets more than one postcard, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- $\vdots$
- n_k possible kth entries for each sequence of first $k-1$ entries,

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k.$$

Counting assignments with restrictions

With the 5 people and 4 postcards case, any of the 5 can be given the first postcard. But, for the second postcard, we cannot reuse whoever got the first one. So, 4 choices left. For the third postcard, there are only 3 choices of people left. And, for the fourth postcard, there are only two people left.

$5 \cdot 4 \cdot 3 \cdot 2 = 120$, which is less than the 625 unrestricted possibilities.

General form, n people, $k < n$ postcards, no one gets more than one: $n \cdot (n-1) \cdot \cdots \cdot (n-k+1)$.

Serial numbers

“Defective” because the serial number (11180916) repeats a digit (1).

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers? 10^8 by the product rule.

How many 8-digit serials numbers have no repeats (not defective)?
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1814400$ by the generalized product rule.

Fraction not defective: not-defective / total = $1814400/100000000 = .018144$ or 1.8%.

Pawn, Bishop, Knight

How many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

There are 8 rows and 8 columns, which we'll number 1 to 8.

For example, P:(1,2), N:(4,1), B:(6,2) is invalid because the pawn and bishop are in the same column (2). Moving B to (6,3) fixes it.

P's row and column are unrestricted: $8 \cdot 8$. N only has 7 remaining choices of row and 7 remaining choices of column: $7 \cdot 7$. Finally, B has 6 choices for each of row and column.

So, $8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 = 112896$ different ways.

Permutations

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

$(w, x, y, z), (w, x, z, y), (w, y, x, z), (w, y, z, x), (w, z, x, y), (w, z, y, x),$
$(x, w, y, z), (x, w, z, y), (x, y, w, z), (x, y, z, w), (x, z, w, y), (x, z, y, w),$
$(y, x, w, z), (y, x, z, w), (y, w, x, z), (y, w, z, x), (y, z, x, w), (y, z, w, x),$
$(z, x, y, w), (z, x, w, y), (z, y, x, w), (z, y, w, x), (z, w, x, y), (z, w, y, x),$

How many permutations on n elements? It's like the non-repeating postcard problem where $n = k$:

$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 = n!.$

Permutation examples

- Assigning n professors n classes to teach (one each).
- Assigning n actors to n roles (one each).
- Visit all of n cities in some order.
- Seating arrangements of n people.
- Ways of shuffling n cards.