Modular Arithmetic, Multiplicative Inverse

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CS 0220 2024

March 4, 2024

Overview

GCD Linear Combination Theorem

Theorem: The greatest common divisor of a and b is a linear combination of a and b. That is, $\gcd(a, b) = s \cdot a + t \cdot b$ for some integers s and t.

Proof: We'll do strong induction on the claim $P(a)$, for all $b \geq a$, $\gcd(a, b) = s \cdot a + t \cdot b$.

Base case: If $a = 0$, $\gcd(a, b) = b = 0 \cdot a + 1 \cdot b$.

Inductive step: Let $b = q \cdot a + r$. Equivalently, $r = 1 \cdot b - q \cdot a$.

$$\begin{aligned}
\gcd(a, b) &= \gcd(r, a) && \text{Remainder thm.} \\
&= s \cdot r + t \cdot a && \text{Inductive hyp.} \\
&= s \cdot (1 \cdot b - q \cdot a) + t \cdot a && \text{Defn of } r \\
&= (t - sq) \cdot a + s \cdot b && \text{Algebra.}
\end{aligned}$$

QED.

Computing the linear combination

We can use this theorem as an algorithm to find the linear combination of a and b that produces their GCD. Returns (s, t, g) where g is the GCD of the input.

def gcdcombo(a, b):
    if $a = 0$: return$(0, 1, b)$
    else:
        $(s, t, g) =$ gcdcombo(rem$(b, a), a)$
        return$(t - s \cdot \text{qcnt}(b, a), s, g)$

- gcdcombo$(0, 15)$ = $(0, 1, 15)$
- gcdcombo$(10, 15)$ = $(-1, 1, 5)$
- gcdcombo$(24, 64)$ = $(3, -1, 8)$

Computing By Hand

a	b	q	s	t	g
24	64				

```
def gcdcombo(a, b):
    if a = 0: return(0, 1, b)
    else:
        (s, t, g) = gcdcombo(rem(b, a), a)
        return(t − s · qcnt(b, a), s, g)
```

Do the rems going down, then the weights going up. Note that, at every level: $sa + tb = g$ (sanity check!).

Computing By Hand

a	b	q	s	t	g
24	64	2	3	-1	8
16	24	1	-1	1	8
8	16	2	1	0	8
0	8		0	1	8

Do the rems going down, then the weights going up. Note that, at every level: $sa + tb = g$ (sanity check!).

Pulvarizing

Corollary: An integer is a linear combination of a and b iff it is a multiple of $\gcd(a, b)$.

Proof (for reference):
Let $g = \gcd(a, b)$. We showed $g = sa + tb$ for some s and t. Any multiple of g is a linear combination of a and b: $kg = k(sa + tb) = (ks)a + (kt)b$.

We know $a = k_1 g$ and $b = k_2 g$ because g is a common divisor of a and b. Any linear combination of a and b is a multiple of g: $s'a + t'b = s'(k_1 g) + t'(k_2 g) = (s'k_1 + t'k_2)g$.

Mixing a and b in different combinations, we can only make multiples of g.

Note: The combinations are not unique: $sa + tb = (s - b)a + (t + a)b$.

Fundamental Theorem of Arithmetic

Theorem: Every integer greater than 1 is a product of a unique non-increasing sequence of primes.

Lemma: If p is a prime and $p|ab$, then $p|a$ or $p|b$.

Proof of Lemma: One case is if $\gcd(a, p) = p$. Then, the claim holds, because a is a multiple of p.

Otherwise, $\gcd(a, p) \neq p$. In this case, $\gcd(a, p)$ must be 1, since 1 and p are the only positive divisors of p. Since $\gcd(a, p)$ is a linear combination of a and p, we have $1 = sa + tp$ for some s, t. Then, $b = s(ab) + (tb)p$; that is, b is a linear combination of ab and p. Since p divides both ab and p, it also divides their linear combination, b. QED.

Proof of Fundamental Theorem of Arithmetic

Lemma: Let p be a prime. If $p|a_1a_2\cdots a_n$, then p divides some a_i.

Proof: Every positive integer can be expressed as a product of primes. (Proved by strong induction!) We need to show this expression is unique. We proceed by contradiction: Assume there exist positive integers that can be written as products of primes in more than one way. Take the smallest such integer n and let $n = p_1p_2\cdots p_j = q_1q_2\cdots q_k$ be the two decompositions. Arrange them in non-increasing order and assume without loss of generality that $p_1 \leq q_1$. If $p_1 = q_1$, the remaining part of the product is smaller than n and different, which is a contradiction (n was the smallest).
Note that all the p_is are less than q_1. But $q_1|n$ and $n = p_1p_2\cdots p_j$, so q_1 divides one of the p_is, which contradicts the fact that q_1 is bigger than all them. QED.

Congruence definition

Definition: a is *congruent to* b modulo n iff $\text{rem}(b, n) = \text{rem}(a, n)$. Equivalently, $n|(a-b)$.

We write $a \equiv b \pmod{n}$.

$29 \equiv 15 \pmod{7}$ because $7|(29-15)$, namely 14. Both have a remainder of 1 when divided by 7.

Equivalence relation—partitions the integers.

Transitivity, reflexivity, symmetry.

Basic modular algebra

In regular algebra,

$$\begin{aligned} a &= b \quad \text{implies} \\ a + c &= b + c. \end{aligned}$$

Can we do the same is congruence-land? $\begin{aligned} a &\equiv b \pmod{n} \\ a + c &\equiv b + c \pmod{n}. \end{aligned}$

Yes!
$a \equiv b \pmod{n}$ iff $n|(a-b)$ iff $\exists k, kn = a - b$ iff $\exists k, kn = a - b + (c - c)$ iff $\exists k, kn = (a+c) - (b+c)$ iff $n|((a+c) - (b+c))$ iff $a + c \equiv b + c \pmod{n}$.

Multiplication is repeated addition, so we can also multiply both sides by a constant. By transitivity, we can always add or multiply each side by values that are congruent! "Clock arithmetic".

Example

$$
\begin{aligned}
2x + 17 &\equiv x + 31 \pmod{12} \\
2x &\equiv x + 14 \pmod{12} && \text{add } -17 \text{ to both sides} \\
2x &\equiv x + 2 \pmod{12} && \text{add 0 to left and } -12 \text{ to right} \\
x &\equiv 2 \pmod{12} && \text{add } -x \text{ to both sides}
\end{aligned}
$$

Double check. $4 + 17 = 21$ vs. 33. Difference is 12, check!

$$
\begin{aligned}
3x + 4 &\equiv 27 \pmod{11} \\
3x &\equiv 23 \pmod{11} && \text{add } -4 \text{ to both sides}
\end{aligned}
$$

Kind of stuck because we don’t (yet) have a “divide both sides by 3” rule.

So, what about division?

If $a \equiv b \pmod{n}$, can we divide both sides by c?

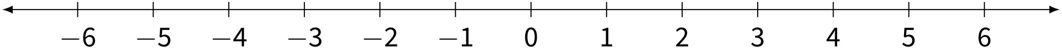
$$\begin{aligned} 7 &\equiv 28 \pmod{3} \\ 1 &\equiv 4 \pmod{3} \quad \text{divide by 7} \end{aligned}$$

So, maybe? At least if the answers are integers?

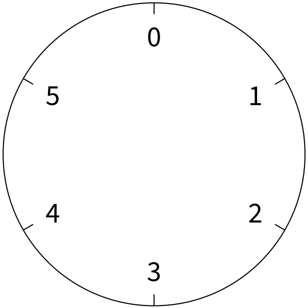
Is division even meaningful more generally?

Equivalent integers or equal “mod-integers”?

We’ve just introduced “equivalence mod n” as a relation on $\mathbb{Z}$.



But we can also think about the “set of integers mod n.”



Equivalent integers or equal “mod-integers”?

What’s the difference? How do we get from one to the other? What structure do they have in common?

For much deeper thoughts here, take a course on *abstract algebra*!