Induction Continued, Strong Induction

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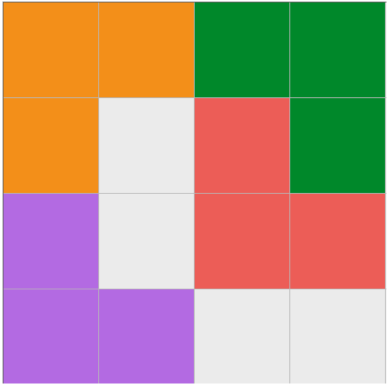
Overview

Definitions

- A *space* is a square.
- A *board* is an $n \times m$ rectangle of spaces with adjacent spaces touching.
- A *tile* is a set of touching squares that can go on top of an identically-shaped set of spaces.
- An *L-shaped tile* is 3 squares where two are touching a central third on consecutive sides.
- A *center space* of a square $n \times n$ board for even n is any of the four spaces touching the midpoint of the board $(n/2, n/2)$.
- A *placement* of tiles on a board assigns each square in a tile to each space on the board so adjacency relationships are maintained and no squares are assigned to the same space. (Injection!)
- A *hole* is a cell not assigned a tile in a given placement.

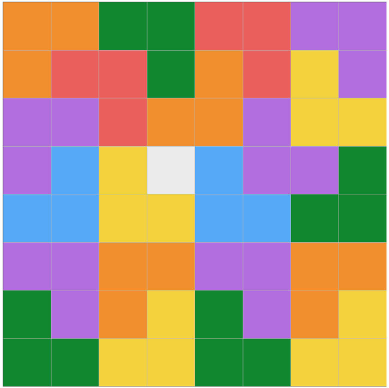
Picture



space? board? tile? L-shared tile? center space? placement? hole?

Theorem

Theorem: For any $n \geq 0$, for any selected space on a $2^n \times 2^n$ board, there exists a tiling of the board using L-shaped tiles such that there is exactly one hole and it is at that selected space.

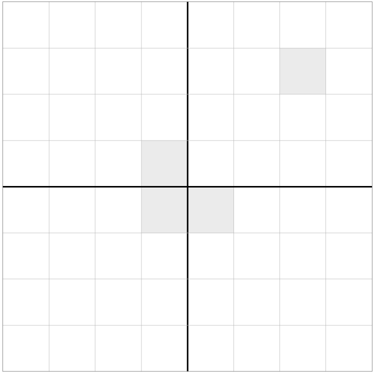
Example: 

Base Case

Proof: The proof is by induction. Let $P(n)$ be the proposition that, for *any* selected space, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on the selected space.

Base case: $P(0)$ is true because there is only one space and there is a tiling (place no tiles) that has exactly one hole and it is in that space.

Inductive Step: Picture



Inductive Step: Words

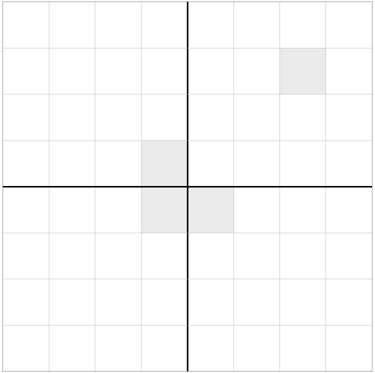
Assume that $P(n)$ is true for some $n \geq 0$; that is, for every space on a $2^n \times 2^n$ board, there exists a placement of L-shaped tiles that leaves only the chosen space empty. Our goal is to show that $P(n+1)$ is true.

Divide the $2^{n+1} \times 2^{n+1}$ board into four quadrants, each $2^n \times 2^n$. One quadrant contains the cell we want to leave empty. Select empty cells for the other three quadrants that are their corners that are center cells for the full board. By four applications of our inductive hypothesis, we can make a placement of L-shaped tiles that leave only those empty cells. Now, add one more L-shaped tile to the board, covering the undesired center cells.

This argument proves that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. Thus, $P(m)$ is true for all $m \in \mathbb{N}$.

Theorem: Repeated

Theorem: For all $n \geq 0$, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on a center space.



Induction starting at a value other than 0

Our previous recipe for induction: to show $P(n)$ for every $n \in \mathbb{N}$, it suffices to show

- $P(0)$
- For an arbitrary n, if $P(n)$ holds, then $P(n+1)$ holds.

Claim: to show $Q(n)$ for every $n \geq k$, it suffices to show

- $Q(k)$
- for an arbitrary $n \geq k$, if $Q(n)$ holds, then $Q(n+1)$ holds.

Proof. Suppose we have a predicate $Q(n)$ with those properties. Define $P(n) = Q(n+k)$. We show $\forall n, P(n)$ holds by induction. $P(0) = Q(0+k)$ holds, by assumption 1. Suppose $P(n)$ holds for an arbitrary n. This means $Q(n+k)$ holds. Since $n+k \geq k$, assumption 2 says $Q(n+k+1)$ holds. And this is $P(n+1)$. So by induction, we have shown $P(n) = Q(n+k)$ holds for all n: that is, $Q(n)$ holds for all $n \geq k$.

Example

Theorem. For every natural number $n \geq 5$, $2^n > n^2$.
Proof. By induction on n starting at 5, with predicate $P(n)$ defined to be $2^n > n^2$.

Base case: $2^5 = 32 > 25 = 5^2$.
Inductive step: suppose $n \geq 5$ and $2^n > n^2$. We want to show $2^{n+1} > (n+1)^2$. Since $n \geq 5$, we have $2n + 1 \leq 3n \leq n^2$. So:

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\leq n^2 + n^2 \\ &< 2^n + 2^n \\ &= 2^{n+1}\end{aligned}$$

This concludes our proof by induction from a starting point.

Strong Induction

Compare these two different "induction step" rules:

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n+1)$.

Strong induction: for all $n \in \mathbb{N}$, $P(0), P(1), \ldots, P(n)$ *together* imply $P(n+1)$.

Strong induction lets you assume more in your inductive step. If you plan to use it in a proof, make sure you say so up front.

A recipe

To prove by strong induction that $P(n)$ holds for every $n \geq k$, we need to show that

- $P(k)$ holds.
- For any $n \geq k$, if $P(k), \ldots, P(n)$ all hold, then $P(n+1)$ holds.

Combines strong induction and induction from a starting point! What happens when $k = 0$?

Example usage of strong induction

Theorem: Every integer greater than 1 is a product of primes.

Our proof will use strong induction on $P(n)$ "n is a product of primes".

We need to prove that $P(n)$ holds for all $n \geq 2$.

Base Case ($n = 2$): $P(2)$ is true because 2 is prime, so it is the product of (one) primes.

Inductive step

Inductive step: Suppose that $n \geq 2$ and that k is a product of primes for every integer k where $2 \leq k \leq n$. We must show that $P(n+1)$, that is, $n+1$ is a product of primes. We proceed by cases:

- $n+1$ is prime: It is the product of (one) primes, so $P(n+1)$ holds.
- $n+1$ is not prime: By definition, $n+1 = km$ for some integers k, m such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both k and m are products of primes. So, therefore, so is km and so is $n+1$. Again, $P(n+1)$ holds.

That completes the cases. And, that completes the strong induction proof.

Another version of strong induction

We can phrase the strong induction principle differently:

To show $P(n)$ holds for all $n \in \mathbb{N}$, it is enough to show that for an arbitrary n, if $P(k)$ holds for every $k < n$, then $P(n)$ holds.

No base case! Why not?

Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: $(0, 1)$.
- 4? No.
- 5? Yes, one 5-unit coin: $(1, 0)$.
- 6? Yes, two 3-unit coins: $(0, 2)$.
- 7? No.
- 8? Yes: $(1, 1)$.
- 9? Yes: $(0, 3)$.
- 10? Yes: $(2, 0)$.
- 11? Yes: $(1, 2)$.

Coin Theorem

Theorem: You can make all values 8 or larger using coins of value 3 and 5.

Proof: We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is n.

Base case: $P(8)$ is true because we can make 8 using $(1, 1)$.

Inductive step: Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n+1)$ holds. We argue by cases:

- $n = 9$: $(0, 3)$
- $n = 10$: $(2, 0)$
- $n \geq 11$: By strong induction, we can make the value $n - 2$, then add a 3-unit coin to get $n + 1$.

That completes the proof by strong induction, which proves the theorem. QED.

Every number is interesting

Theorem. Every natural number is interesting.

Proof. We proceed by strong induction with the property $P(n)$ = "n is interesting."

Base case: 0 is certainly interesting!

Inductive case: suppose $0, \ldots, n$ are all interesting. If $n + 1$ were not interesting, it would be the smallest noninteresting number. But that's interesting!

The game

Start with a list of numbers and a score of zero. Pick an integer z from the list and replace it with two positive integers x and y such that $x + y = z$. Add xy to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

7							0
5					2		10
2		3			2		16
2		2		1	2		18
1	1	2		1	2		19
1	1	2		1	1	1	20
1	1	1	1	1	1	1	21

Irrelevance theorem

Theorem: No matter how you play the game starting from $n \geq 1$, your score will be $n(n-1)/2$.

Proof: The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number n gives a score of $n(n-1)/2$.

Base case: Starting from a list of $n = 1$, the game ends immediately with a score of $n(n-1)/2 = 1(0)/2 = 0$. $P(1)$ is true.

Inductive step:We must show that $P(1), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 1$. Assume that $P(1), \ldots, P(n)$ are all true and we are playing starting with $n+1$. The first move must break $n+1$ into positive numbers a and b with $a + b = n + 1$. The total score for the game is the sum of points for this first move plus the scores obtained from playing with a and b:

Inductive step

total score	=	score for turning $n+1$ into a and b	
		+ score from starting with a	
		+ score from starting with b	rules of game
	=	$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2}$	inductive hyp.
	=	$\frac{2ab+a^2-a+b^2-b}{2}$	
	=	$\frac{(a+b)^2-(a+b)}{2}$	algebra
	=	$\frac{(n+1)^2-(n+1)}{2}$	Defn a, b
	=	$(n+1)n/2$	algebra

We showed $P(1), P(2), \ldots, P(n)$ implies $P(n+1)$, showing the claim true by strong induction. QED.

Proving strong induction

We can *prove* that this principle of strong induction works, using regular induction. Suppose: for all $n \in \mathbb{N}$, if $P(k)$ holds for every $k < n$, then $P(n)$ holds. (Call this hypothesis H.)

We'll proceed by induction on the property $Q(n) =$ "$P(k)$ holds for every $k < n$."

Base case: $Q(0)$ holds, since $P(k)$ holds for every $k < 0$.

Inductive step: Suppose $Q(n)$ holds. So $P(k)$ holds for every $k < n$. By our hypothesis H, $P(n)$ holds. Thus, $P(k)$ holds for every $k < n + 1$, which is $Q(n + 1)$.

Strong Induction vs. Induction vs. Well Ordering (5.3)

Covered in the book, Chapter 2.

Idea: If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn't the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we're showing that being true up to some value n means it can't be false for n.

And, strong induction is just ordinary induction with a universal quantifier in its key proposition.

So, three different names for the same thing. Use is a matter of taste.