Induction

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Overview

Touching base

We've talked about logic, sets, functions, relations, and Lean. Why?

Big questions when writing a proof:

- Where do I start?
- What proof method do I use?
- Is my proof a good proof?

Emphasis of the first part of this class: *information management* and *shape of propositions.*

Lean: makes these informal ideas into a rigorous game. Also: can apply these ideas to programming!

Paying it forward

We're all friendly at the CS22 cafe, where there's a long line for coffee. I (an observer not in line) want to get a "pay it forward" chain going:

1. I buy coffee for the person at the front of the line (position 0).
2. Each person in line (at position n) buys coffee for the next person (at position $n + 1$) if their own coffee was paid for.

Who gets coffee bought for them?

Everyone. Why? Because the pay-it-forward scheme "propogates" down the line until everyone is included.

What if we only included the first rule? Only the 0th person is guaranteed to get coffee.

What if we only included the second rule? No one would be guaranteed to get coffee.

Infinite coffee

What if the line at the cafe was *infinite* and we followed the same two rules?

Would the zero-th person get coffee? Yes, rule one.

Would the first person get coffee? Yes, rule one and rule two.

Would the second person get coffee? Yes, rule one and rule two twice.

Would the nth person get coffee for $n \in \mathbb{N}$? Yes, rule one and rule two n times.

Would the ∞ person get a ticket? No, ∞ is not a number.

Even though the *process* never ends, we can say that everyone gets a coffee. (The students deep in the line might have to wait, but they *will* get one.)

The Principle of Induction

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- for every nonnegative integer n, if $P(n)$ is true, then $P(n+1)$ is true,

then

- $P(m)$ is true for every nonnegative integer m.

It's a great way to prove a *global* property $\forall n \in \mathbb{N}, P(n)$ from two *local* properties: $P(0) \wedge (\forall n \in \mathbb{N}, P(n) \rightarrow P(n+1))$

Bowling pins

Theorem: For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Written using *sigma/summation* notation:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof: Define $P(n)$ to be the claim that the sum of the numbers from 1 to n is $n(n+1)/2$. We want to show that for every $n \in \mathbb{N}$, $P(n)$ is true.

$P(0)$ is true because the sum of no numbers is zero and $0 \cdot 1/2$ is also zero.

Bowling pins, Induction Step

Now, we need to show that for some arbitrary n, *if* $P(n)$ is true, *then* $P(n+1)$ must be true. So, consider what happens when $P(n)$ is true. That means the sum of the numbers from 1 to n is $n(n+1)/2$ for that n. (We're not assuming it's true for all n.) We want to show that the sum from 1 to $n+1$ is $(n+1)(n+2)/2$.

$$\begin{aligned} &1+2+3+\ldots+(n+1) \\ &\quad = (1+2+3+\ldots+n)+(n+1) \\ &\quad = n(n+1)/2+(n+1) \\ &\quad = (n+1)(n/2+1) \\ &\quad = (n+1)(n+2)/2 \end{aligned}$$

That means $P(n+1)$ is true. Since we have $P(0)$ and also $P(n)$ implies $P(n+1)$, we have $\forall m \in \mathbb{N}, P(m)$ by induction. QED.

Bowling pins, Observations

- What we showed is that the amount that $n(n+1)/2$ increases when n increases by 1 is exactly $n+1$.
- You can think of it as a summary of checking $P(0)$, $P(1)$, $P(2)$, $P(3)$ and then getting bored and giving a recipe that lets us check any other P we want on the fly.
- Kind of like recursion in programming?
- In Lean: what would the proof state look like at various points of this argument?

A template

1. State that the proof uses induction.
2. Make the appropriate induction hypothesis $P(n)$ explicit.
3. Prove that the base case $P(0)$ is true.
4. Prove the induction step: $P(n)$ implies $P(n+1)$ for every nonnegative integer n.
5. Invoke induction and conclude that $P(n)$ is true for all nonnegative n.

Once more, condensed

Proof: We use induction. The induction hypothesis, $P(n)$, is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{1}$$

Base case: $P(0)$ is true, because both sides of Equation 1 equal zero when $n = 0$.

Inductive step: Assume that $P(n)$ is true, where n is any nonnegative integer. Then,

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) && \text{(unfolding the sum)}\\
&= \frac{n(n+1)}{2} + (n+1) && \text{(by induction hypothesis)}\\
&= \frac{(n+1)(n+2)}{2}, && \text{(by simple algebra)}
\end{aligned}$$

which proves $P(n+1)$.

So, it follows by induction that $P(n)$ is true for all nonnegative n. QED.

When proofs go bad

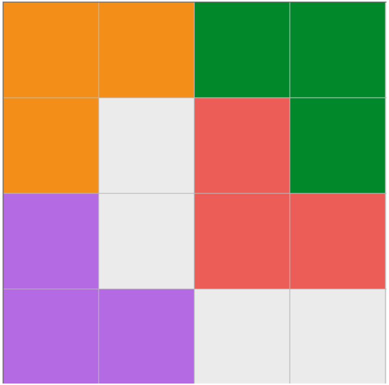
There's a number of ways an induction proof can go off the rails.

- Didn't prove base case. Every number n equals $n+1$. If $P(n)$ is true, $n=n+1$. Adding one to both sides shows $P(n+1)$, $n+1=n+2$. But, $P(0)$ is not true!
- Didn't really prove the inductive step. There's always room for more students in CS0220. There's room for zero. If we can fit n people, we can always squeeze in one more. Not really.
- Proved the base case *and* inductive step, but not for all $n \geq 0$. All horses are the same color. (For every set of n horses, all the horses in that set are the same color.) True for 0 horses. Suppose it's true for n, and give me a set of $n+1$ horses. The first n horses are all the same color and the last n horses are all the same color. So they're all the same color. Doesn't hold for $n+1=2$.

Definitions

- A *space* is a square.
- A *board* is an $n \times m$ rectangle of spaces with adjacent spaces touching.
- A *tile* is a set of touching squares that can go on top of an identically-shaped set of spaces.
- An *L-shaped tile* is 3 squares where two are touching a central third on consecutive sides.
- A *center space* of a square $n \times n$ board for even n is any of the four spaces touching the midpoint of the board $(n/2, n/2)$.
- A *placement* of tiles on a board assigns each square in a tile to each space on the board so adjacency relationships are maintained and no squares are assigned to the same space. (Injection!)
- A *hole* is a cell not assigned a tile in a given placement.

Picture

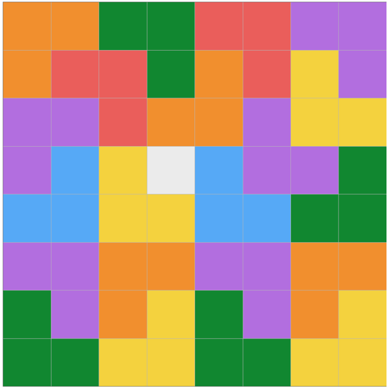


space? board? tile? L-shared tile? center space? placement? hole?

Theorem

Theorem: For any $n \geq 0$, for any selected space on a $2^n \times 2^n$ board, there exists a tiling of the board using L-shaped tiles such that there is exactly one hole and it is at that selected space.

Example:

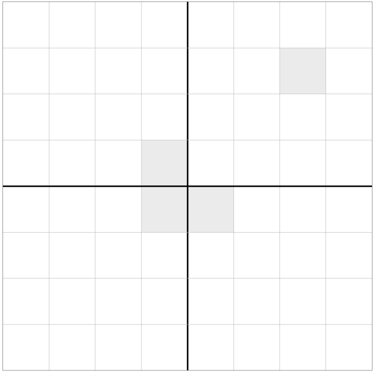


Base Case

Proof: The proof is by induction. Let $P(n)$ be the proposition that, for *any* selected space, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on the selected space.

Base case: $P(0)$ is true because there is only one space and there is a tiling (place no tiles) that has exactly one hole and it is in that space.

Inductive Step: Picture



Inductive Step: Words

Assume that $P(n)$ is true for some $n \geq 0$; that is, for every space on a $2^n \times 2^n$ board, there exists a placement of L-shaped tiles that leaves only the chosen space empty. Our goal is to show that $P(n+1)$ is true.

Divide the $2^{n+1} \times 2^{n+1}$ board into four quadrants, each $2^n \times 2^n$. One quadrant contains the cell we want to leave empty. Select empty cells for the other three quadrants that are their corners that are center cells for the full board. By four applications of our inductive hypothesis, we can make a placement of L-shaped tiles that leave only those empty cells. Now, add one more L-shaped tile to the board, covering the undesired center cells.

This argument proves that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. Thus, $P(m)$ is true for all $m \in \mathbb{N}$.

Theorem: Repeated

Theorem: For all $n \geq 0$, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on a center space.