Properties of Relations

Robert Y. Lewis

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Overview

Relation properties warmup

Reminder from Wed:

partial function: [$\leq$ 1 out]. surjective: [$\geq$ 1 in]. total: [$\geq$ 1 out]. injective: [$\leq$ 1 in]. bijective: [$=$ 1 out] and [$=$ 1 in].

Which property does the following formula describe?
$\forall a : A, \forall b_1 : B, \forall b_2 : B, (a\ R\ b_1 \land a\ R\ b_2) \to b_1 = b_2$

Functions

Let R be a relation with domain A and codomain B. If

$\forall a : A, \forall b_1 : B, \forall b_2 : B, (a \; R \; b_1 \land a \; R \; b_2) \to b_1 = b_2$

then R is a partial function.

If it's also the case that

$\forall a \in A, \exists b \in B, a \; R \; b,$

then R is a function.

We often write functions as $f(a)$, where $f : A \to B$, meaning we give f an $a \in A$ and it returns the $b \in B$ such that $a \; f \; b$.

Function image

Suppose $f : A \to B, X \subseteq A$.

The image $f(X)$ is $\{y \in B \mid \exists x \in X, f(x) = y\}$. All the elements of the codomain that are "mapped to" from some element of X.

The image of the entire domain, $f(A)$, is called the *range* of f.

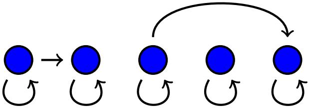
When is the range of f the same as the codomain of f?

If f is surjective, $f(A) = B$.

Reflexivity

Let R be a binary relation on A.

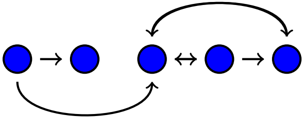
Definition: R is *reflexive* iff $\forall a \in A, a\ R\ a$.



Irreflexivity

Let R be a binary relation on A.

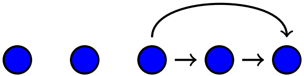
Definition: R is *irreflexive* iff $\forall a \in A, a \not R\ a$. (Not the same as not reflexive!)



Transitivity

Let R be a binary relation on A.

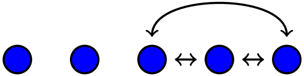
Definition: R is *transitive* iff $\forall a, b, c \in A, (a\ R\ b \wedge b\ R\ c) \rightarrow a\ R\ c$.



Symmetry

Let R be a binary relation on A.

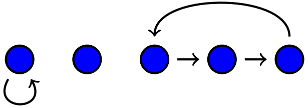
Definition: R is *symmetric* iff $\forall a, b \in A, a \; R \; b \rightarrow b \; R \; a$.



Antisymmetry

Let R be a binary relation on A.

Definition: R is *anti-symmetric* iff $\forall a \neq b \in A, a\ R\ b \to b\ \not R\ a$. (Not the same as not symmetric!)



RST Examples

In a group of people:

- reflexive and symmetric but not transitive: “is standing closer than 6 feet away from”
- reflexive and transitive but not symmetric: “is no taller than”
- transitive and symmetric but not reflexive: “is in CS0220 with” (assuming some people are not in CS0220).

Can you think of more examples of relations that are

- reflexive and symmetric but not transitive?
- reflexive and transitive but not symmetric?
- transitive and symmetric but not reflexive?

All of the above

Definition: A relation that is simultaneously reflexive, symmetric, and transitive is an *equivalence* relation.

The relation *partitions* the domain.

Example: “rounds to the same value as”.

- Reflexive: x rounds to the same value as x.
- Symmetric: If x rounds to the same value as y, then y rounds to the same value as x.
- Transitive: If x rounds to the same value as y, and y rounds to the same value as z, then x must round to the same value as z.

We’re really just restating key properties of the equality relation.

Equivalence classes

If R is an equivalence relation on A, we can partition the elements of A into sets $[a]_R = \{a' \in A | a \; R \; a'\}$.

Here, the a in $[a]_R$ is an arbitrarily chosen representitive of its *equivalence class*.

Example: Define relation R on $\mathbb{Z}$ by $x \; R \; y$ iff $x \bmod 3 = y \bmod 3$.

Then, we have:

- $[0]_R = \{\ldots, -3, 0, 3, 6, \ldots\}$
- $[1]_R = \{\ldots, -2, 1, 4, 7, \ldots\}$
- $[2]_R = \{\ldots, -1, 2, 5, 8, \ldots\}$

These three sets are a *partition* of the integers. That means mutually exclusive and exhaustive. That means $[0]_R \cup [1]_R \cup [2]_R = \mathbb{Z}$, $[0]_R \cap [1]_R = \emptyset$, $[0]_R \cap [2]_R = \emptyset$, and $[1]_R \cap [2]_R = \emptyset$.

Properties of elements of sets

Some more philosophical ruminations, if we have time in class today!

One reason sets are useful in mathematics is they let us infer properties of individuals from properties of sets:

- All positive integers can be expressed as the product of powers of primes.
- 20527342709 is a positive integer.
- So, 20527342709 can be expressed as the product of powers of primes.

Intersections of sets

Continues to hold under intersection.

- $\forall x \in S_1, P_1(x)$.
- $\forall x \in S_2, P_2(x)$.
- $a \in S_1 \cap S_2$.
- So, $P_1(a) \wedge P_2(a)$.

We can characterize an element by the properties of the sets it is in.

Gaining Properties

Is it necessarily the case that *a only* has the properties of the sets that contain it? Yes, but that's vacuous. After all, $\{a\}$ is the set with just a in it so a may have some properties completely unique to a (like being a!) shared with elements of that set and no other elements.

Less trivial example:

- $\forall x \in S_1, x$ is divisible by three.
- $\forall x \in S_2, x$ is divisible by two (even).
- $\forall x \in S_1 \cap S_2, x$ is divisible by two AND x is divisible by three.
- But also, $\forall x \in S_1 \cap S_2, x$ is divisible by six.

Being in the intersection caused it to gain a new property.

Exists Properties are Weaker

Things are more complex when the properties of the set are less absolute.

- $\forall x \in S_1, P_1(x)$.
- $\exists x \in S_2, P_2(x)$.
- $a \in S_1 \cap S_2$.
- $P_1(a)$? Yes! $P_2(a)$? Don't know.

Or even:

- $\exists x \in S_1, P_1(x)$.
- $\exists x \in S_2, P_2(x)$.
- $a \in S_1 \cap S_2$.
- $P_1(a)$? Don't know. $P_2(a)$? Don't know.

Intersectionality

When these sets are sets of people, we need to be very careful about jumping to conclusions.

That's one lesson that comes from the concept of "intersectionality"—people in the intersection of multiple groups necessarily share some properties with the groups they belong to (the "for all" properties), but they can also gain new properties from being in the interesection. And they can also lose weaker properties (the "exists" properties).

We'll return to this idea when we discuss statistical properties of sets later in the semester.