Functions, Injectivity, Surjectivity, Bijections

Robert Y. Lewis

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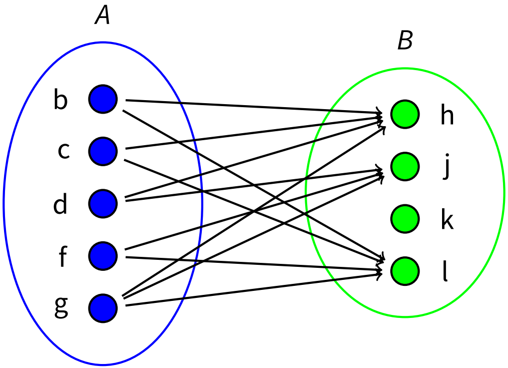
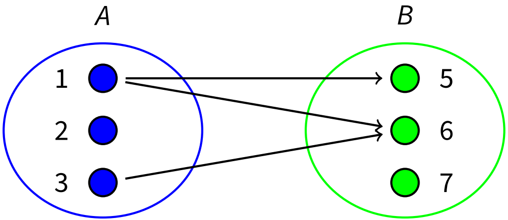
Overview

Binary relations

Definition. A *binary relation*, R, consists of a set, A, called the domain of R, a set, B, called the codomain of R, and a subset of $A \times B$ called the graph of R.



Properties of relations

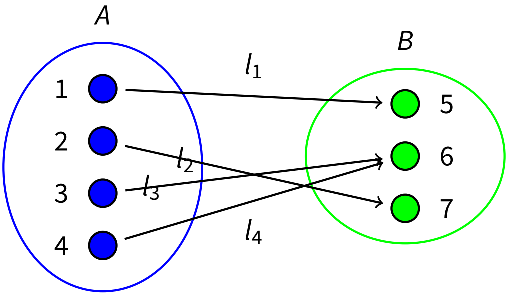


A binary relation:

- is a *partial function* when it has the [$\leq$ 1 arrow out] property.
  Book: “function”. Us: “function” is [$=$ 1 arrow out] property.
- is *surjective* when it has the [$\geq$ 1 arrows in] property.
- is *total* when it has the [$\geq$ 1 arrows out] property.
- is *injective* when it has the [$\leq$ 1 arrow in] property.
- is *bijective* when it has both the [$=$ 1 arrow out] and the [$=$ 1 arrow in] properties.

Example relation #1

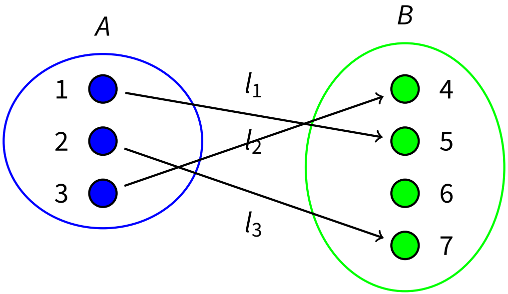
partial function: [$\leq$ 1 out]. surjective: [$\geq$ 1 in]. total: [$\geq$ 1 out]. injective: [$\leq$ 1 in].
bijective: [$=$ 1 out] and [$=$ 1 in].



Partial function; surjective; total. Not injective, not bijective.
Summary: a surjective function. (Implies partial function and total.)

Example relation #2

partial function: [$\leq$ 1 out]. surjective: [$\geq$ 1 in]. total: [$\geq$ 1 out]. injective: [$\leq$ 1 in].
bijective: [$=$ 1 out] and [$=$ 1 in].

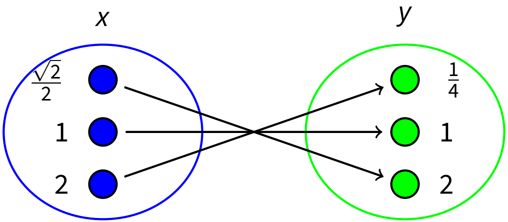


Partial function; total; injective. Not surjective, not bijective.
Summary: an injective function. (Implies partial function and total.)

Example relation #3

partial function: [$\leq$ 1 out]. surjective: [$\geq$ 1 in]. total: [$\geq$ 1 out]. injective: [$\leq$ 1 in]. bijective: [$=$ 1 out] and [$=$ 1 in].

Equation $y = 1/x^2$ on $\mathbb{R}^+$. x is an element in the domain, y is an element in the co-domain.

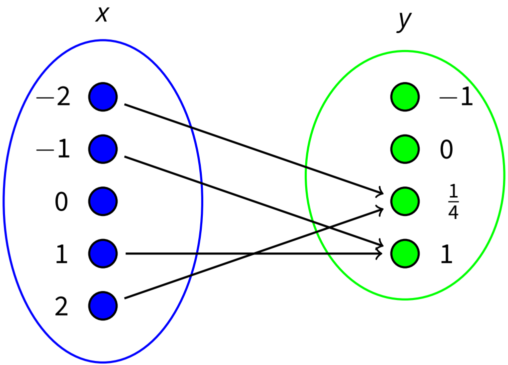


Partial function; surjective; total; injective; bijective.
Summary: a bijective (partial) function. (Implies everything else.)

Example relation #4

partial function: [$\leq$ 1 out]. surjective: [$\geq$ 1 in]. total: [$\geq$ 1 out]. injective: [$\leq$ 1 in]. bijective: [$=$ 1 out] and [$=$ 1 in].

Equation $y = 1/x^2$ on $\mathbb{R}$.



Partial function. Not anything else.

Image definition

Definition. The *image* of a set $Y \subseteq A$ under a relation $R : A \to B$, written $R(Y)$, is the subset of elements of the codomain B of R that are related to some element in Y.

In terms of the relation diagram, $R(Y)$ is the set of points with an arrow coming in that starts from some point in Y.

$R(Y) = \{x \in B \mid \exists y \in Y, y \; R \; x\}$.

Inverse definition

Definition: The *inverse* R^{-1} of a relation $R : A \to B$ is the relation from B to A defined by the rule

$b \; R^{-1} \; a \leftrightarrow a \; R \; b.$

Definition: The image of a set under the relation R^{-1} is called the *inverse image* of the set. That is, the inverse image of a set X under the relation R is defined to be $R^{-1}(X)$.

Example: $x \; R \; y$ iff there's a dictionary word with first letter x and second letter y. The image $R(\{c, k\})$ is the letters that can appear after 'c' or 'k' at the beginning of a word. It's the set $\{a, e, h, i, l, n, o, r, u, v, w, y, z\}$.

The inverse image $R^{-1}(\{c, k\})$ is the letters that can appear before 'c' or 'k' at the beginning of a word. It's the set $\{a, e, i, o, s, t, u\}$.

Inverses of relations

What can we infer about R^{-1} if R is:

- partial function? injective
- surjective? total
- total? surjective
- injective? partial function
- bijective? bijective
- function? injective and surjective

More examples to consider

Can you come up with examples of relations on $\mathbb{R}$ that are:

- Surjective, not a partial function?
- A partial function, total, injective but not surjective?
- Everything (a bijective function)? (Something different from $y = x$!)