♡ Relations ♡

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Overview

Definition

Definition: A *sequence* is an ordered list of items.

Like a set: Collection of items.

Unlike a set: Order matters, can have repeats.

Simple example: Cartesian coordinates like (x, y).

Definition: The *empty sequence* is a sequence of length 0 and is denoted λ.

Operation on sets: Product

Definition: The product of two sets A and B is written $A \times B$ and is the set of all length-two sequences where the first element of the sequence comes from A and second from B.

Example:

- $X = \{1, 2\}$
- $Y = \{0, 2, 4\}$
- $X \times Y = \{(1, 0), (1, 2), (1, 4), (2, 0), (2, 2), (2, 4)\}$
- $|X \times Y| = 6 = |X| \times |Y|$

Extends to chains of products: $S_1 \times S_2 \times S_3 \times \cdots S_k$.

If $S_1 = S_2 = S_3 = \cdots = S_k = S$, can write simply S^k.

Example: $\{a, b\}^3 = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b),$
$(b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$.

Binary relations, intuitively

Binary here just means “two”. Relation means we’re talking about a relationship between two items.
Examples:

- “less than”, $a < b$ defines a relationship between two items. The relationship *holds* if and only if a represents a smaller number than b. $5 < 11$, $2021 \not< 1984$.
- “subset”, $A \subseteq B$.
- “is the square of”.
- “is a factor of”.

Binary relations, formally

Definition. A *binary relation*, R, consists of a set, A, called the domain of R, a set, B, called the codomain of R, and a subset of $A \times B$ called the graph of R.

Kind of like a two-place predicate.
If both the domain and codomain are the same set A, we say it's a relation "on A".

Infix notation is common: $a \ R \ b$, meaning (a, b) is in the graph of R.

Binary relation examples

Binary relations are essentially functions that take in two items and return a Boolean.

superset ($\supset$)?	Yes.
add ($+$)?	No, binary, but returns a number.
equals ($=$)?	Yes.
is divisible by?	Yes.
element of ($\in$)?	Yes. Domain and codomain can be different.
is a perfect square?	No, only takes one input. It's a predicate.
is the perfect square of?	Yes.
greatest common divisor?	No, returns a number.
greatest common divisor is one?	Yes, also called "relatively prime" or "co prime".
or?	Sure, you could see it that way.

Other binary relations and problems

- Is married to?
- Is taller than?
- Is older than?
- Happened before?
- Has met?
- Is faster than?
- Is smarter than?
- Lives with?
- Speaks the same languages as?
- Speaks a same language as?

A philosophical question (again)

We’re once again introducing new vocabulary.

Are sets defined in terms of something else? Or is “set” an atomic concept, and we just describe how sets work?

What about sequences?

What about relations?

A philosophical question (again)

The answer depends on our choice of foundations.

These concepts are all interrelated. In some contexts, sets are primitive and relations are defined. In others, this is flipped.

Pragmatically: does it matter? As long as we (the communicators) *agree*, it doesn't change what we do or say.

Defining ordered pairs

We just defined the term “binary relation” in terms of sets: it’s a subset of a product of sets, containing ordered pairs.

But what’s an ordered pair, really?

$(a, b) \neq \{a, b\}$, since $\{a, b\} = \{b, a\}$ but $(a, b) \neq (b, a)$.

Defining ordered pairs

We define the ordered pair (a, b) in terms of sets:

$$(a, b) = \{a, \{a, b\}\}$$

Claim: for any sets X, Y, it cannot be the case that $X \in Y$ and $Y \in X$.

(Why?)

Defining sequences

Once we have ordered pairs, (finite) sequences are easy: just keep pairing.

$$
\begin{aligned}
(a, b, c) &= ((a, b), c) \\
&= \{(a, b), \{(a, b), c\}\} \\
&= \{\{a, \{a, b\}\}, \{\{a, \{a, b\}\}, c\}\}
\end{aligned}
$$