Expectation and Variance

Recitation 10

Expected Value

Intuitively, the expected value is the weighted average of values, kind of a mass center of the probability distribution.

More formally, the *expected value* of a random variable is denoted $\mathbb{E}[X]$ and is defined as

$$\mathbb{E}[X] = \sum_{s \in S} X(s) \Pr(s) = \sum_{r \in X(S)} r \Pr(X = r).$$

We define the *conditional expected values* as follows: Given that event E has occurred, the expectation of random variable X is

$$\mathbb{E}[X \mid E] = \sum_{r \in X(S)} r \Pr(X = r \mid E). \tag{1}$$

Moreover, the *linearity of expectation* can be very useful in calculating expected value: Given that Z, X, Y are three random variables defined on a sample space S and a and b are two real numbers such that $Z = aX + bY$, we know that $\mathbb{E}[Z] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ must be true.

Let's practice this through a task:

Task 1

Ilija and Justin are playing a game. They flip a fair coin 3 times. When the coin is heads, Justin pays $1 to Ilija; and when the coin is tails, Ilija pays $1 to Justin.

a. Let G_i be a random variable representing what Ilija gains on the i-th round. For instance, $G_3 = -1$ if the coin is tails.

What is the expected value of G_i?

Solution:

Since $\Pr(G_i = 1) = \frac{1}{2}$ and $\Pr(G_i = -1) = \frac{1}{2}$, $\mathbb{E}[G_i] = \frac{1}{2} - \frac{1}{2} = 0$.

b. Let G be a random variable that represents Ilija's *total* gain in this game. What is the expected value of G?

> **Solution:**
>
> $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \mathbb{E}[G_3]$, where G_i is the expected gain from flip i. We already know that $\mathbb{E}[G_i] = 0$, so $\mathbb{E}[G] = 0 + 0 + 0 = 0$ as well.

c. What is the expected value of G if the coin is biased and the probability of heads is p? in other words, generalize your solution from part b in terms of p.

> **Solution:**
>
> Now, $\mathbb{E}[G_i] = p \cdot 1 + (1-p) \cdot (-1) = p - 1 + p = 2p - 1$. Thus, $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \mathbb{E}[G_3] = (2p-1) + (2p-1) + (2p-1) = 6p - 3$.

d. Ilija and Justin are still using the biased coin from part c. Let H_1 be the event that the first coin is heads. What is $\mathbb{E}[G|H_1]$?

> **Solution:**
>
> $\mathbb{E}[G \mid H_1] = 1 + (2p-1) + (2p-1) = 4p - 1$.
>
> G_1 is 1 from the first round guaranteed, and the other terms are from part c.

e. Use your answers to calculate $\mathbb{E}[G]$ and $\mathbb{E}[G \mid H_1]$ when $p = 0.7$.

> **Solution:**
>
> $\mathbb{E}[G] = 6p - 3 = 6 \cdot 0.7 - 3 = 1.2$
>
> $\mathbb{E}[G \mid H_1] = 4p - 1 = 4 \cdot 0.7 - 1 = 1.8$

f. Assume that $p = 0.7$ and let's say we want to change the game to make it "fair." If the flip is tails, then Ilija pays a dollar to Justin—how much should Justin pay Ilija on Heads so that for any number of flips we know $\mathbb{E}[G] = 0$?

> **Solution:**
>
> Each $\mathbb{E}[G_i]$ is $0.3 \cdot (-1) + 0.7 \cdot x$, where x is the number of dollars that Justin should pay Ilija on Heads. Thus, we have $\mathbb{E}[G] = (0.3 \cdot (-1) + 0.7 \cdot x) \cdot 3 = 0$. Solving for x, we have $x = \frac{3}{7}$. So, Justin should pay Ilija \$0.43 on heads so that the game is fair.

Task 2

Ilija and Justin are now playing a similar, but different, game. This time they flip a coin **2 times**. Let X be the random variable that is equal to the number of heads and Y the random variable that is equal to the number of tails. At the end of the game, Justin pays Ilija X^2 dollars. Once again, let G be the random variable for Ilija's gain.

a. Calculate $\mathbb{E}[X]^2$ when the probability of heads is $p = 0.7$.

Solution:

$\mathbb{E}[X] = 0.7 + 0.7 = 1.4$. So, $\mathbb{E}[X]^2 = 1.4^2 = 1.96$.

b. Calculate $\mathbb{E}[G]$ when $p = 0.7$.

Solution:

$$\begin{aligned}\mathbb{E}[G] &= \mathbb{E}[X^2] \\ &= (0.7)^2(2)^2 + \binom{2}{1}(0.3)(0.7)(1^2) + (0.3)^2(0)^2 \\ &= 1.96 + 0.42 \\ &= 2.38\end{aligned}$$

There is a $(0.7)^2$ chance of flipping a heads both times which leads to a payoff of 2^2, a $\binom{2}{1}(0.3)(0.7)$ chance of flipping a heads and tails which leads to a payoff of 1^2, and a $(0.3)^2$ chance of flipping a tails both times which leads to a payoff of 0.

c. Does $\mathbb{E}[G] = \mathbb{E}[X]^2$?

Solution:

No!

In general for any random variable with nonzero variance we have $\mathbb{E}[X^2] > \mathbb{E}[X]^2$.

d. Find an example that shows that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ does not hold where X and Y are not independent variables.

Hint: Try X and Y as defined above.

Solution:

Assume we have a fair coin. We know that X and Y, as defined in this problem, are not independent variables. We know that $\mathbb{E}[X]\mathbb{E}[Y] = 1 \cdot 1 = 1$. We can also calculate $\mathbb{E}[XY]$ to be $(\frac{1}{4})(2)(0) + (\frac{1}{2})(1)(1) = (\frac{1}{4})(0)(2) = \frac{1}{2}$.

Thus, this is an example of where X and Y are dependent variables where

$\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$.

e. Prove that if X and Y are two independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Solution:

Assume X and Y are independent. So for any x and y in their range we have $\Pr(X = x \wedge Y = y) = \Pr(X = x)\Pr(Y = y)$. We now use the definition of expected value.

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_{x \in X(S)} \sum_{y \in Y(S)} xy \Pr(X = x \wedge Y = y) && (2)\\
&= \sum_{x \in X(S)} \sum_{y \in Y(S)} xy \Pr(X = x)\Pr(Y = y) && (3)\\
&= \left(\sum_{x \in X(S)} x \Pr(X = x) \right) \left(\sum_{y \in Y(S)} y \Pr(Y = y) \right) && (4)\\
&= \mathbb{E}[X]\mathbb{E}[Y] && (5)
\end{aligned}$$

Bayes' Rule

Bayes' Rule can be summarized as

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

where A and B are events and $\Pr(B) \neq 0$.

Task 3

Assume Brown's CS department has an evaluation system for CS courses based on student evaluations. In any class, the students can fill out the evaluation form and give a score of $0, 1$, or 2 to the course. Let X be the random variable of this score. The students of CS0220 either like the course with probability $3/4$ (Event L) or they do not like the course with probability $1/4$ (Event $\neg L$).

Assume that the conditional probability distribution of X given L is

$$\Pr(X = 0 \mid L) = 1/8$$

$$\Pr(X = 1 \mid L) = 1/4$$
$$\Pr(X = 2 \mid L) = 5/8$$

and given that they do not like the course ($\neg L$) it is

$$\Pr(X = 0 \mid \neg L) = 9/10$$

$$\Pr(X = 1 \mid \neg L) = 1/10$$
$$\Pr(X = 2 \mid \neg L) = 0.$$

a. If a student has given a score of 0 to CS0220, what is the probability that they do not like the course?

Solution:

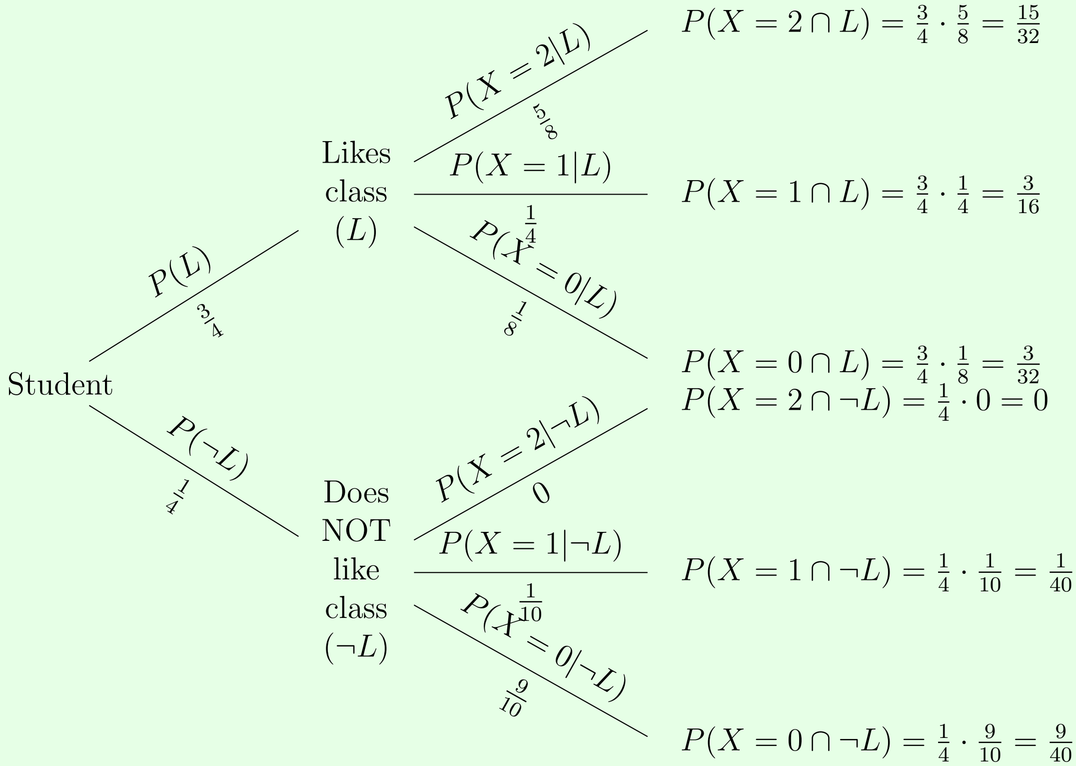
Using Bayes' Rule,

$$\begin{aligned}
\Pr(\neg L \mid X = 0) &= \frac{\Pr(X = 0 \mid \neg L)\Pr(\neg L)}{\Pr(X = 0)} \\
&= \frac{\Pr(X = 0 \mid \neg L)\Pr(\neg L)}{\Pr(X = 0 \cap L) + \Pr(X = 0 \cap \neg L)} \\
&= \frac{\Pr(X = 0 \mid \neg L)\Pr(\neg L)}{\Pr(X = 0|L)P(L) + \Pr(X = 0|\neg L)P(\neg L)}
\end{aligned}$$

$$= \frac{\frac{9}{10} \cdot \frac{1}{4}}{\frac{3}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{9}{10}}$$
$$= \frac{12}{17}$$

The formula for conditional probability $\Pr(\neg L \mid X = 0) = \frac{\Pr(\neg L \cap X=0)}{\Pr(X=0)}$ gives the same answer.

Note: Drawing out a diagram like this may be helpful for this problem:



b. Use the definition of conditional expected value (Equation 1) to find $\mathbb{E}[X \mid \neg L]$

Solution:

$\mathbb{E}[X \mid \neg L] = \sum r \Pr(X = r \mid \neg L) = 2 \cdot 0 + 1 \cdot \frac{1}{10} + 0 \cdot \frac{9}{10} = \frac{1}{10}$.

c. *Optional:* Find $\mathbb{E}[X]$.

Solution:

$$\mathbb{E}[X] = 2\Pr(X = 2) + 1\Pr(X = 1) + 0\Pr(X = 0)$$

$$
\begin{aligned}
=&2\left(\Pr(X=2 \mid L)\Pr(L) + \Pr(X=2 \mid \neg L)\Pr(\neg L)\right) \\
&+ \Pr(X=1 \mid L)\Pr(L) + (\Pr(X=1 \mid \neg L)\Pr(\neg L) \\
&= 2 \cdot \frac{3}{4} \cdot \frac{5}{8} + 1 \cdot \frac{3}{4} \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} \cdot \frac{1}{10} = \frac{23}{20} = 1.15.
\end{aligned}
$$

Alternative solution: $\mathbb{E}[X] = \Pr(L)\mathbb{E}[X \mid L] + \Pr(\neg L)\mathbb{E}[X \mid \neg L] = \frac{3}{4} \cdot (\frac{5}{8} \cdot 2 + \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot 0) + \frac{1}{4} \cdot (0 \cdot 2 + \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot 0) = \frac{23}{20} = 1.15$

Checkpoint #1 — Call over a TA!

Variation from the mean

Sometimes measuring the mean (expectation) of a random variable doesn't give us enough information: it can be helpful to know how much we expect the variable to *stray* from its average.

Definition. The *variance* $\text{Var}[R]$ of a random variable R is defined to be $\mathbb{E}[(R - \mathbb{E}[R])^2]$.

Unpacking this from the inside out: $R - \mathbb{E}[R]$ is a random variable measuring the distance between R and its mean at each outcome. Averaging the square of this gives us a sense of, overall, how far R tends to be from its mean.

There is an equivalent way to state this:

Lemma. For any random variable R,

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2.$$

Task 4

Prove the above lemma.

Solution:

For notational convenience, we will define $\mu := \mathbb{E}[R]$.

$$\begin{aligned}\text{Var}[R] &= \mathbb{E}[(R-\mu)^2] \\ &= \mathbb{E}[R^2 - 2\mu R + \mu^2] \\ &= \mathbb{E}[R^2] - 2\mu\mathbb{E}[R] + \mu^2 \\ &= \mathbb{E}[R^2] - 2\mu^2 + \mu^2 \\ &= \mathbb{E}[R^2] - \mu^2 \\ &= \mathbb{E}[R^2] - (\mathbb{E}[R])^2\end{aligned}$$

Note that the third line follows by linearity of expectation, with $\mathbb{E}[\mu^2] = \mu^2$ since μ is a constant.

Markov's inequality gives a generally coarse estimate of the probability that a random variable takes a value much larger than its mean.

Theorem (Markov). If R is a nonnegative random variable, then for all $x > 0$,

$$\Pr[R \geq x] \leq \frac{\mathbb{E}[R]}{x}.$$

Expressed differently:

Corollary. If R is a nonnegative random variable, then for all $c \geq 1$,

$$\Pr[R \geq c \cdot \mathbb{E}[R]] \leq \frac{1}{c}.$$

That is: the probability of R being more than c times its mean is at most $1/c$.

This leads us to state Chebyshev's theorem, an application of Markov's inequality:

Theorem (Chebyshev). Let R be a random variable and $x \in \mathbb{R}^+$. Then

$$\Pr\left[|R - \mathbb{E}\left[R\right]| \geq x\right] \leq \frac{\text{Var}[R]}{x^2}.$$

Task 5

Suppose you flip a fair coin 100 times. The coin flips are all mutually independent.

a. What is the expected number of heads?

> **Solution:**
>
> Let X be the random variable denoting the number of heads.
>
> $\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{100}] = \underbrace{(1 \cdot \frac{1}{2}) + (1 \cdot \frac{1}{2}) + \cdots (1 \cdot \frac{1}{2})}_{\text{100 times}} = 100 \cdot \frac{1}{2} = 50$

b. Using Markov's inequality, what upper bound can we place on the probability that the number of heads is at least 70?

> **Solution:**
>
> $\Pr[X \geq 70] \leq \frac{\mathbb{E}[X]}{70} = \frac{50}{70}$
>
> Using Markov's inequality, we can derive a corresponding upper bound of $\frac{5}{7} \approx 0.714$.

c. What is the variance of the number of heads? The following theorem may help:

> **Theorem.** Let X and Y be independent random variables. Then
>
> $$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].$$

(*Note*: This does **not** hold as a general property of variance!)

Solution:

Given that all coin flips are mutually independent, we can say

$$\mathrm{Var}[X] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots \mathrm{Var}[X_{100}].$$

For X_i such that $1 \leq i \leq 100$,

$$\mathrm{Var}[X_i] = \mathbb{E}[(X_i)^2] - (\mathbb{E}[X_i])^2.$$

Calculating the terms independently, we get

$$\mathbb{E}[(X_i)^2] = (1)^2 \cdot \frac{1}{2} + (0)^2 \cdot \frac{1}{2} = \frac{1}{2}$$

$$(\mathbb{E}[X_i])^2 = (\frac{1}{2})^2 = \frac{1}{4}$$

So

$$\mathrm{Var}[X_i] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Since this generally holds for the i^{th} flip, we can conclude

$$\mathrm{Var}[X] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots \mathrm{Var}[X_{100}] = \underbrace{\frac{1}{4} + \frac{1}{4} + \cdots \frac{1}{4}}_{\text{100 times}} = 25.$$

d. Using Chebyshev's Theorem, what upper bound does can we place on the probability that the number of heads is either less than 30 or greater than 70?

Solution:

$\Pr[X > 70 \cup X < 30] = \Pr[|X - 50| \geq 20] = \Pr[|X - \mathbb{E}[X]| \geq 20]$

By Chebyshev's Theorem, $\Pr[|X - \mathbb{E}[X]| \geq 20] \leq \frac{25}{20^2} = \frac{1}{16}$.

Thus, we are given an upper bound of $\frac{1}{16} = 0.0625$.

Task 6

A cactarium of cacti is stricken by a blizzard. The blizzard lowers a cactus's temperature from normal levels, and a cactus will die if its temperature goes below 90 degrees F. The blizzard is so intense that it lowered the average temperature of the cactarium to 85 degrees. Temperatures as low as 70 degrees, but no lower, were actually found in the cactuarium.

a. Use Markov's inequality to prove that at most 3/4 of the cacti are expected to survive.

> *Hint*: If you run into issues with directly applying Markov's inequality to the expected value of the temperature itself, think about how we can manipulate the inequality $R \geq x$ to express the bound differently – perhaps in relation to the minimum temperature. (If $R \geq x$, then $R - y \geq x - y$.)

Solution:

Let random variable T denote a cactus's temperature in degrees F.

$\mathbb{E}[T]$ is the average temperature of the cactarium, so $\mathbb{E}[T] = 85$

Applying Markov's inequality to T, we get

$$\Pr[T \geq 90] \leq \frac{\mathbb{E}[T]}{90} = \frac{85}{90}$$

.

But $\frac{85}{90} = \frac{17}{18} > \frac{3}{4}$, so this is not tight enough of a bound.

Instead, apply Markov's inequality to $T - 70$:

$$\Pr[T \geq 90] = \Pr[T - 70 \geq 20] \leq \frac{\mathbb{E}[T-70]}{20} = \frac{85-70}{20} = \frac{15}{20} = \frac{3}{4}.$$

b. Suppose there are 400 cacti in the cactarium. Give an example of a set of temperatures for the cacti so that the average cactarium temperature is 85 and 3/4 of the cacti will have a high enough temperature to survive. Deduce that the bound from part a is the best possible.

Solution:

Let 300 cacti have a temperature of 90 degrees and the remaining 100 have a temperature of 70 degrees. The average temperature of the cactarium is $\frac{300 \cdot 90 + 100 \cdot 70}{400} = \frac{34000}{400} = 85$ degrees.

Checkoff — Call over a TA!