Recitation 4

Relations and Functions

Part 1: Relations

Definitions

Defn 1: A *binary relation* $R : A \to B$ is defined on a domain A, co-domain B, and a graph that is a subset of the Cartesian product $A \times B$ (called the graph of R). If $R : A \to B$, we say R maps A to B.

Given $a \in A$ and $b \in B$, if $R(a, b)$ is true then we say 'a is related to b by R', and write $a\ R\ b$, meaning (a, b) is in the graph of R.

Defn 2: A relation *on* a set A is $R : A \to A$.

Defn 3: An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

Defn 4: A relation R on A is *reflexive* if $\forall a \in A$, $(a, a) \in R$.

Defn 5: A relation R on A is *symmetric* if $\forall (a, b) \in R$, $(b, a) \in R$. An equivalent definition is that a relation is *not* symmetric if $\exists (a, b) \in R$ such that $(b, a) \notin R$.

Defn 6: A relation R on A is *antisymmetric* if $\forall a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$.

Defn 7: A relation R on A is *transitive* if $\forall (a, b), (b, c) \in R$, $(a, c) \in R$. An equivalent definition is that a relation is *not* transitive if $\exists (a, b), (b, c) \in R$ such that $(a, c) \notin R$.

Defn 8: Let R be an equivalence relation on A. Then, the *equivalence class* of $a \in A$, denoted $[a]_R$, is $\{a \in A \mid a\ R\ a\}$. That is, $[a]_R$ is all of the elements to which a is related.

Equivalence Relations

What does it mean for two things to be essentially "the same"? It can depend on context. Say you're trying to organize your closet by color. While a red shirt and red boots are distinct items, we may want to consider them as similar for the sake of organization. The relation here is sharing the same color.

Let's run through the intuition of what makes this an equivalence relation:

- Reflexive: An item has the same color as itself. For example, a pink bow is "related" to itself because it is pink (same color).

- Symmetric: If item one is the same color as item two, then item two is also the same color as item one. For example, if a yellow blouse is the same color as a yellow skirt, then we know that the yellow skirt is the same color as the yellow blouse.
- Transitive: If item one is the same color as item two, and item two is the same color as item three, then item one must be the same color as item three. For example, if a pair of blue jeans is the same color as a blue dress, and a blue dress is the same color as a blue shirt, then blue jeans are the same color as a blue shirt.

Equivalence relations give us a way of saying that two elements of a set are 'similar', without having to be equal.

An equivalence relation partitions the elements of the domain into sets — these sets are the equivalence classes! In the context of the previous example, the domain (your closet) is partitioned into equivalence classes (the sets of items that share the same color). All red items form an equivalence class, all blue items form another, and so forth.

Formally, a *partition* of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

As we formalize these concepts mathematically, it will be useful to refer back to the rigorous definitions above to prove these three properties.

Task 1

1. Consider the set $A = \{1, 2, 3\}$. In the following questions, all relations are on A. It may be helpful to draw out a diagram of each relation.

   a. $R = A \times A$. List out the elements of R. Is R an equivalence relation? If so, state its equivalence class(es).

   > $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$. It is an equivalence relation because everything is related to each other, since it's the entire Cartesian product. There is only one equivalence class: the entire set.

   b. $R = \{(1,2),(2,1)\}$. Is this relation *transitive*?

> No, it's not transitive. $(1,2),(2,1) \in R \implies (1,1) \in R$, but it's not actually in R. The same argument works for $(2,2)$.

c. $R = \{(1,2),(2,1),(2,2),(1,1)\}$. Is this relation reflexive? Symmetric? Transitive?

> $(3,3)$ is not in R, so it's not reflexive (since we defined R to be on A. However, it is transitive and symmetric.

d. If the relation in question iii is not an equivalence relation, can you add one pair to it and make it an equivalence relation? Write the equivalence classes of the new relation.

> Yes, add $(3,3)$. In that case, the equivalence classes become $\{1,2\}$ and $\{3\}$.

2. Let $A = \{1,2\}$ and answer to the following questions.

a. What is the equivalence relation on A with the smallest number of equivalence classes possible?

> $A \times A$, a world where everything is equal to everything.

b. What is the equivalence relation on A with the largest number of equivalence classes possible?

> $\{(1,1),(2,2)\}$, where things are equal only if they are truly equal.

c. Is $R_0 = \{\}$ a relation on A?

> Yes, the empty set is a subset of every set

d. Is R_0 symmetric? Is it antisymmetric? Why or why not?

> Yes, it does not violate our definition of symmetry/antisymmetry as there are no pairs to begin with. ("vacuously true" based on logical equivalence $(F \implies P) \equiv T$)

e. Is R_0 transitive? Why or why not?

> Yes, it does not violate our second definition of transitive as there are no pairs. ("vacuously true based on the logical equivalence $(F \implies P) \equiv T$")

f. R_0 is not an equivalence relation. Why?

> It is not reflexive.

3. Suppose R is an equivalence relation on S, and $R = \{\}$. What is S?

> The empty set, otherwise it is not reflexive.

4. Consider the set B of all students at Brown. For each of the following relations on B, state whether they are reflexive, symmetric, antisymmetric, transitive, or some combination of them. If it is an equivalence relation, then determine the equivalence classes of the relation.

a. Two students are related if they have the same astrology sign.

> Reflexive, symmetric, and transitive. Therefore equivalence relation. Equivalence classes are students of same astrology sign.
>
> However, not antisymmetric.

b. s_1 and s_2 are students and $(s_1, s_2) \in R$ if s_1 is younger than or the exact same age as s_2. (You can assume no students were born at the exact same time.)

> Transitive and anti-symmetric and reflexive, but not symmetric.

c. Two students are related if they are studying anthropology.

> Symmetric and transitive but not reflexive or anti-symmetric.

d. Two students are related if they go to Brown.

> Reflexive, symmetric, and transitive, but not antisymmetric. Therefore equivalence relation. One equivalence class which consists of all students at Brown.

Checkpoint 1 — Call a TA over!

Part 2: Functions

Definitions

Defn 1: A relation $R : X \to Y$ is a **function** if for every x in the domain X, x is mapped to one and only one y in Y, the codomain. Note that in the book this is called a *total function*, and function refers to a *partial function*, where for every x in the domain X, x is mapped to zero or one y in the codomain Y. In this class, we will use function to mean total function and partial function to mean partial function.

Defn 2: The **range** of a function f consists of all members of the codomain of f that are mapped to by some member of the domain of f. It is the *image* of the domain.

Defn 3: $f : X \to Y$ is **injective (one-to-one)** if, for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently, for any $x, y \in Y$ we have $f(x) = f(y) \implies x = y$, and you can also use its contrapositive $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Defn 4: $f : X \to Y$ is **surjective (onto)** if, for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. For surjective functions, the range is equal to co-domain.

Defn 5: $f : X \to Y$ is a **bijection** if it is both an injection and surjection.

Task 2

Let A be the set $\{1, 2, 3\}$. Consider the following relation on A, $R_1 = \{(1, 2), (2, 1)\}$.

1. Is R_1 a function?

> No; not all members of A are mapped to something in A. It is a partial function but not a (total) function.

Now, consider R_2, another relation on A: $\{(1, 2), (2, 1), (3, 2)\}$.

1. Is R_2 a function?

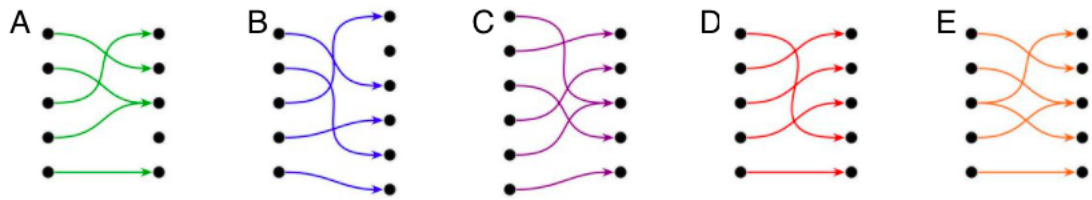
> Yes.

2. If R_2 is a function, what is its codomain? How about its range?

> The codomain is A. The range is $\{1, 2\}$.

Task 3

Consider these diagrams that visualize a relation $R : A \to B$. The diagrams have two sets of dots, one for A and one for B, and they have an arrow from a to b in whenever $(a, b) \in R$.



Match each of the five diagrams, labeled A–E, with one of these five descriptions below:

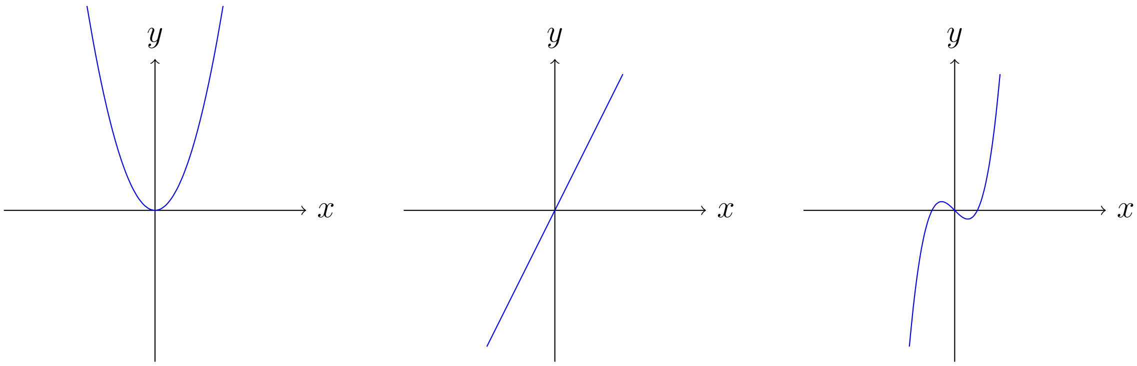
1. ___ Not a function
2. ___ A function that is neither surjective nor injective
3. ___ A surjective function that is not injective
4. ___ An injective function that is not surjective
5. ___ A bijective function — both surjective and injective

> 1. E is not a function
> 2. A is a function that is neither surjective nor injective
> 3. C is a surjective function that is not injective
> 4. B is an injective function that is not surjective
> 5. D is a bijective function

Optional **Checkpoint (recommended if queue is short) — Call a TA over!**

Task 4

Consider the following functions and determine if the given function is an injection, surjection, and/or bijection.



(a) $f(x) = x^2$ (b) $g(x) = x/2$ (c) $h(x) = x^3 - x$

Discuss your answers!

a. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

> Not injective or surjective.

b. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{x}{2}$

> Injective and surjective. Therefore bijective.

c. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^3 - x$

> Surjective, but not injective

d. Question: All of the above functions are defined on $\mathbb{R}$. Consider their graphs in the coordinate system. Which of the following implies surjectivity, which implies injectivity, and which implies neither?

(1) any vertical line intersects the graph **at most** once

(2) any horizontal line intersects the graph **at most** once

(3) any vertical line intersects the graph **at least** once

(4) any horizontal line intersects the graph **at least** once

> Surjection: (4). Injection: (2). Neither: (1), (3)

e. Recall the functions defined in parts a-c. Is $f \circ h : \mathbb{R} \to \mathbb{R}$ surjective and/or injective? (Use a graphing calculator if you need to.)

> $f \circ h(x) = (x^3 - x)^2$, not surjective, not injective

f. Is $h \circ f : \mathbb{R} \to \mathbb{R}$ surjective and/or injective?

> $h \circ f(x) = x^6 - x^2$, not surjective, not injective

g. Let $f : \mathbb{R} \to \mathbb{Z}$ give as output the greatest integer less than or equal to x, denoted as the floor function $f(x) = \lfloor x \rfloor$. For instance, $f(3.5) = 3$, $f(3) = 3$, and $f(\pi) = 3$.[1]

> Surjective, but not injective.

h. $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = \lfloor x \rfloor$

> Injective and surjective. Therefore bijective.

i. *Optional:* f : Brown University Students $\to$ Countries in the World

f(student) = country where student is from

> Not injective, and sadly not surjective.

j. *Optional:* f : First Year Students $\to$ First Year Dorms

f(student) = dorm that student lives in

> Not injective. Presumably surjective.

[1] Note that we can similarly define the ceiling function $f : \mathbb{R} \to \mathbb{Z}$ that gives as output the smallest integer greater than or equal to x, denoted the ceiling function $f(x) = \lceil x \rceil$. For example, $hf(3.5) = 4$, $f(3) = 3$, and $f(\pi) = 4$.

Task 5

Given sets A and B, a function $f : A \to B$ is *injective* if

$$\forall a, b \in A,\ f(a) = f(b) \to a = b.$$

(This is a formalization of the intuitive "arrow counting" definition given in lecture.)

Given sets A, B, and C, and functions $g : B \to C$ and $f : A \to B$, we define the *composition* of g and f, written $g \circ f : A \to C$, by $(g \circ f)(x) = g(f(x))$ for all $x \in A$. In other words, to apply $g \circ f$ to an argument x, first apply f to x, and then apply g to the result. (Beware the order of operations!)

Let S be a set and $f : S \to S$ be a function. Prove that f is injective if and only if $f \circ f$ is injective.

Once you're done, pick a point a few sentences into your proof. (Don't pick the very beginning or very end.) Sketch out the "proof state" at this position, informally like you might see in Lean: what are the *hypotheses* in your context at this point? What is your goal? (Do you have any additional goals? If so, what are the contexts there?)

Fix a set S and let $f : S \to S$ be a function. We show both directions of the claim.

First, suppose f is injective; we will show $f \circ f$ is injective. Let $x, y \in S$ and suppose that $(f \circ f)(x) = (f \circ f)(y)$. By the definition of composition, we have $f(f(x)) = f(f(y))$. (*) By the injectivity of f, it follows that we have equality of inputs $f(x) = f(y)$; by the injectivity of f once more, we then have $x = y$. So $f \circ f$ is injective.

Now suppose $f \circ f$ is injective; we show f is injective. Let $x, y \in S$ such that $f(x) = f(y)$. Then observe that $f(f(x)) = f(f(y))$ by the uniqueness property of functions. This is equivalent to $(f \circ f)(x) = (f \circ f)(y)$ by the definition of composition, so $x = y$ follows by the injectivity of $f \circ f$. So f is injective, as desired. $\square$

As an illustration of the "proof state" part, we sketch what we might see at (*). There are two goals and their associated contexts:

1.
$S : Set$
$f : S \to S$
$hf : \forall a, b \in A,\ f(a) = f(b) \to a = b$
$x, y : S$
$hfxy : f(f(x)) = f(f(y))$

$\vdash x = y$ (the goal)

2.
$S : Set$
$f : S \to S$
$hf : \forall a, b \in A,\ f(f(a)) = f(f(b)) \to a = b$
$\vdash \forall a, b \in A,\ f(a) = f(b) \to a = b$ (the goal)

Optional **Task 6**

a. If a function $f : X \to Y$ is injective, what can we say about the cardinalities of X and Y? Try making some diagrams where X has more elements than Y, fewer elements than Y, or the same number of elements as Y. When are you able to create an injection, and when are you not?

> If and only if $|X| \leq |Y|$, then there exists $f : X \to Y$ such that f is injective.

b. If a function $f : X \to Y$ is surjective, what can we say about the cardinalities of X and Y? Again, you might want to draw out some examples.

> If and only if $|X| \geq |Y|$, then there exists $f : X \to Y$ such that f is surjective.

c. Based on what you've found in the previous two questions, if a function $f : X \to Y$ is bijective, what can we say about the cardinalities of X and Y? When can we create a bijection between two sets, and when can we not?

> If and only if $|X| = |Y|$, then there exists $f : X \to Y$ such that f is bijective.

Checkoff - Call a TA over!