Recitation 3

Set Theory and Set Algebra

Set Theory

Defn 1: A **set** is a collection of objects with no repetition or order.

Defn 2: B is a **subset** of A if every element in B is also in A. This is written as $B \subseteq A$.

Defn 3: The **integers** are the set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. The **non-negative integers** (also called the natural numbers) are the set $\mathbb{N} = \{0, 1, 2, ...\}$.

Defn 4: A number n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. A number n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Defn 5: A number n is **rational** if $n = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, where $b \neq 0$. The set of rational numbers is denoted as $\mathbb{Q}$.

Defn 6: $\mathcal{P}(A)$, called the **power set** of A, is the set of all subsets of A.

Defn 7: $A \cup B$ denotes the **union** of sets A and B. This contains all the elements from A, and all of the elements from B.

Defn 8: $A \cap B$ denotes the **intersection** of sets A and B. This contains only the elements that appear in both of the sets.

Defn 9: $A \setminus B$ denotes the **difference** of sets A and B. This contains elements that appear in A but not B.

Defn 10: $\overline{A}$ denotes the **complement** of A relative to some universal set U. $\bar{A} = U \setminus A$, that is, it is everything except what is in A.

Defn 11: $|A|$ denotes the **cardinality** of A, which is a count of the number of elements contained in A.

Defn 12: The symbol $\forall$ means **for all**.

Defn 13: The symbol $\exists$ means **there exists**.

Task

a. True or False: A is an arbitrary set. Answer true only if the statement is always true. That is, answer true only if for any possible set A, the statement is true.

   i. $A \subseteq A$
   ii. $\{\} \subseteq A$
   iii. $\{\} \in A$

   > i. True
   > ii. True
   > iii. False

b. True or False: $\mathbb{N} \subseteq \mathbb{Z}$

   > True

c. $\{0, 1, 9\} \subseteq \mathbb{N}$

   > True

d. True or False: $\{-1.5, 9\} \subseteq \mathbb{Z}$

   > False

e. i. $\mathbb{Q} \cap \mathbb{N} = \mathbb{N}$
   ii. $\mathbb{Q} \cup \mathbb{N} = \mathbb{R}$

   > i. True
   > ii. False

f. S is the set of flying dinosaurs that lived during the Cretaceous period. G is the set of all dinosaurs that lived during the Cretaceous period. The Pterodactyl was a flying dinosaur that lived during the Cretaceous period.

   True or False:

   i. $S \subseteq G$
   ii. Pterodactyl $\subseteq S$
   iii. Pterodactyl $\in S$
   iv. $\{\text{Pterodactyl}\} \subseteq G$

i. True
ii. False
iii. True
iv. True

g. True or False: If $A = \{1, 2, 4\}$ then $\{2, 4\} \in \mathcal{P}(A)$

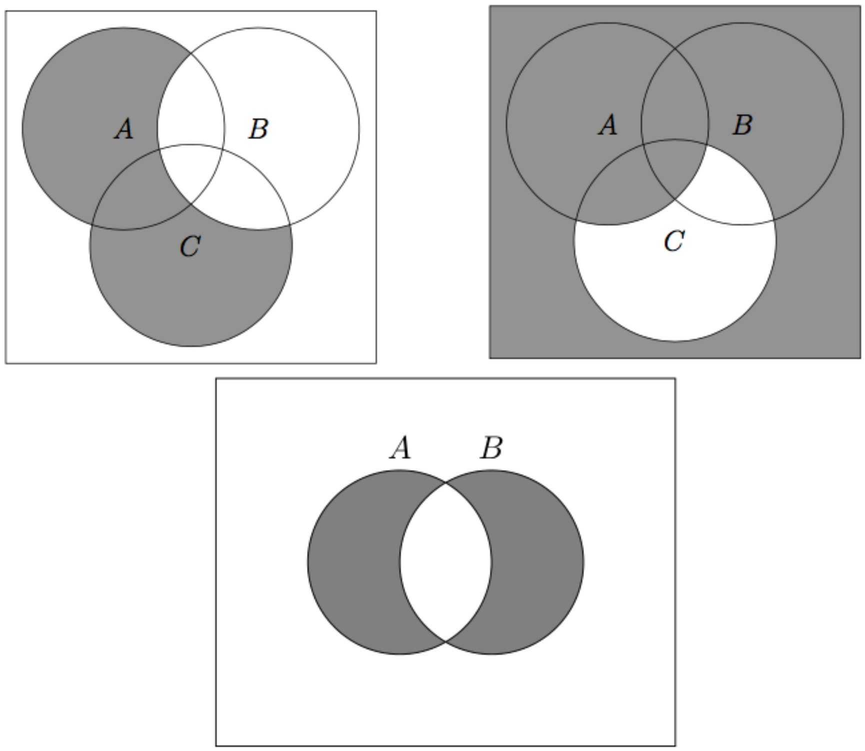
True

h. True or False: A is a set, and $\mathcal{P}(A)$ is the set of all subsets of A. Answer true only if the statement is always true.

i. $A \in \mathcal{P}(A)$
ii. $A \subseteq \mathcal{P}(A)$
iii. $\emptyset \in \mathcal{P}(A)$
iv. $\emptyset \subseteq \mathcal{P}(A)$

i. True
ii. False
iii. True
iv. True

i. In each of the following Venn diagrams, A, B, and C are sets and are assumed to be subsets of a universal set (denoted by the rectangle). Write a set algebraic expression (i.e. one involving union, intersection, difference, and complement) in terms of A, B, and C for each shaded in region.



$(A \cup C) \setminus B$
$\overline{C} \cup A$
$(A \cup B) \setminus (A \cap B)$

j. *Optional:* Call a the cardinality of A and b the cardinality of B. Call s the cardinality of $A \cap B$. For the third picture, what is the cardinality of the set formed from the expression you derived?

$a + b - 2s$

Task 1

Define the following sets

$$A = \{\varnothing, 0, \{\varnothing\}, \{0, \varnothing\}\}$$

$$
\begin{aligned}
B &= \{\varnothing, \{\varnothing\}, \{0, \varnothing, 1\}\} \\
C &= \{x \mid \exists\, y \in \mathbb{Z} \text{ s.t. } y^2 = x,\ x < 10\} \\
D &= \{a, b, c, d, e\} \\
E &= \{c, d, e\} \\
F &= \{d, e, f, g\}
\end{aligned}
$$

a. Find the following sets:

i. $A \cup B$

ii. $A \cap B$

iii. $A \setminus B$

iv. $\mathcal{P}(B)$ $(= 2^B)$, the power set of B

v. $C \setminus (A \cup B)$

vi. $\{x \mid x \in B, |x| \notin C\}$

vii. $A \times E$

viii. *Optional:* $(D \times E) \setminus (E \times F)$

ix. *Optional:* $(D \setminus E) \cap (D \cap E)$

> i. $\{\varnothing, 0, \{\varnothing\}, \{0, \varnothing\}, \{0, \varnothing, 1\}\}$
>
> ii. $\{\varnothing, \{\varnothing\}\}$
>
> iii. $\{0, \{0, \varnothing\}\}$
>
> iv. $\Big\{ \varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{0, \varnothing, 1\}\},$
> $\{\varnothing, \{\varnothing\}\}, \{\varnothing, \{0, \varnothing, 1\}\},$
> $\{\{\varnothing\}, \{0, \varnothing, 1\}\},$
> $\{\varnothing, \{\varnothing\}, \{0, \varnothing, 1\}\} \Big\}$
>
> v. $\{1, 4, 9\}$
>
> vi. $\{\{0, \varnothing, 1\}\}$
>
> vii. $\{(\varnothing, c), (\varnothing, d), (\varnothing, e),$
> $(0, c), (0, d), (0, e),$
> $(\{\varnothing\}, c), (\{\varnothing\}, d), (\{\varnothing\}, e),$
> $(\{0, \varnothing\}, c), (\{0, \varnothing\}, d), (\{0, \varnothing\}, e)\}$
>
> viii. $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, c), (d, c), (e, c)\}$

ix. $\varnothing$

b. Find the cardinalities of the following sets:

i. C

ii. $A \times B$

iii. $\mathcal{P}(A \cup B)$

iv. $F \setminus (D \cap E)$

v. *Optional:* $\mathcal{P}(\mathcal{P}(\varnothing))$

i. 4

ii. 12

iii. 32

iv. 2

v. 2

Checkpoint 1 — Call over a TA!

Set Equivalence

Set Element Method

Defn 1: Two sets S and T are equal if and only if they have the same elements:

$$S = T \iff (\forall x : x \in S \iff x \in T)$$

Defn 2: S and T are equal if and only if both S is a subset of T and T is a subset of S.

From these two definitions, we can construct our set-element method for proving set equalities; that is, we show that two sets are equal if and only if every element of S is also an element of T and every element of T is also an element of S.

In other words, to prove that one set is a subset of the other, we can show that $S \subseteq T$ for arbitrary sets S and T using the following steps:

1. Let x be an element of S.
2. Prove that x is an element of T.
3. Conclude that $S \subseteq T$.

Example

Claim: $A \cap (A \cup B) = A$.

Proof: We show that both $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$.

1. We will first prove our sub-claim that $A \cap (A \cup B) \subseteq A$.

   Consider any $x \in A \cap (A \cup B)$. This means that $x \in A$ and $x \in A \cup B$. In particular, we know that $x \in A$. Thus, we have shown that $A \cap (A \cup B) \subseteq A$.

2. We will then prove the second sub-claim that $A \subseteq A \cap (A \cup B)$.

   Consider any $x \in A$. Then, $x \in A \cup B$. Thus, we know that $x \in A$ and $x \in A \cup B$, which can be rewritten as $x \in A \cap (A \cup B$. This shows that $A \subseteq A \cap (A \cup B)$.

Therefore, $A = A \cap (A \cup B)$.

Now, let's practice!

Task 2

For each of these statements, either prove (using the 'set element' method) or disprove the claim.

Hint: Use a counterexample to disprove a claim!

a. For any two sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

> *Claim*: For all sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
>
> *Proof*:
>
> We will use the set element method to prove our claim. We will first show $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, and we will then show $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.
>
> Consider some x in $\mathcal{P}(A \cap B)$. If x is in $\mathcal{P}(A \cap B)$, then x must be a subset of $A \cap B$. By the definition of intersection, x is a subset of A and x is a subset of B. This means x must be a member of $P(A)$ and x must be a member of $P(B)$. Therefore, x must be a member of $P(A) \cap P(B)$. Thus, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
>
> Now, consider some y in $\mathcal{P}(A) \cap P(B)$. By the definition of intersection, y is in both $\mathcal{P}(A)$ and $\mathcal{P}(B)$. If y is in both $\mathcal{P}(A)$ and $\mathcal{P}(B)$, then y is a subset of both A and B. Thus, $\mathcal{P}(A \cap B)$ must also contain y. Thus, $\mathcal{P}(A) \cap P(B) \subseteq \mathcal{P}(A \cap B)$.
>
> Because $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ and $\mathcal{P}(A) \cap P(B) \subseteq \mathcal{P}(A \cap B)$, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap P(B)$.

b. For any two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

> To disprove our claim, it suffices to provide a counterexample. Let A be $\{1\}$. Let B be $\{3\}$.
>
> $\mathcal{P}(A \cup B) = \{\{1\}, \{3\}, \{1, 3\}, \varnothing\}$.
>
> $\mathcal{P}(A) = \{\{1\}, \varnothing\}$.
>
> $\mathcal{P}(B) = \{\{3\}, \varnothing\}$.
>
> Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{3\}, \varnothing\}$, which is not $\{\{1\}, \{3\}, \{1, 3\}, \varnothing\}$. Thus $\mathcal{P}(A \cup B)$ is not equal to $\mathcal{P}(A) \cup \mathcal{P}(B)$.

Task 3

a. Let A and B be subsets of some universal set. Then $A \setminus (A \setminus B) = A \cap B$.

> Let A and B be subsets of some universal set. We will prove that $A \setminus (A \setminus B) = A \cap B$ by proving that $A \setminus (A \setminus B) \subseteq A \cap B$ and that $A \cap B \subseteq A \setminus (A \setminus B)$.
>
> 1. We will show that $A \setminus (A \setminus B) \subseteq A \cap B$.
>
>   Let $x \in A \setminus (A \setminus B)$. This means that $x \in A$ and $x \notin (A \setminus B)$.
>
>   We know that an element is in $(A \setminus B)$ if an only if it is in A and not in B. Since $x \notin (A \setminus B)$, we conclude that $x \notin A$ or $x \in B$. However, we also know that $x \in A$ and so we conclude that $x \in B$. This proves that $x \in A$ and $x \in B$.
>
>   This means that $x \in A \cap B$, and hence we have proved that $A \setminus (A \setminus B) \subseteq A \cap B$.
>
> 2. We will show that $A \cap B \subseteq A \setminus (A \setminus B)$.
>
>   Now, we choose $y \in A \cap B$. This means that $y \in A$ and $y \in B$.
>
>   We note that $y \in (A \setminus B)$ if and only if $y \in A$ and $y \notin B$ and hence, $y \notin (A \setminus B)$ if and only if $y \notin A$ or $y \in B$. Since we know that $y \in B$, we conclude that $y \notin (A \setminus B)$. Since $y \in A$ and $y \notin (A \setminus B)$, we can conclude that $y \in A \setminus (A \setminus B)$.
>
>   This proves that if $y \in A \cap B$, then $y \in A \setminus (A \setminus B)$ and hence, $A \cap B \subseteq A \setminus (A \setminus B)$.
>
> Since we have proved that $A \setminus (A \setminus B) \subseteq A \cap B$ and $A \cap B \subseteq A \setminus (A \setminus B)$, we conclude that $A \setminus (A \setminus B) = A \cap B$.

b. *Optional:* Let A and B be the following sets:

$$\begin{aligned} A =& \{6n : n \in \mathbb{Z}\} \\ B =& \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \end{aligned}$$

Prove that $A = B$.

> We will show that both $A \subseteq B$ and $B \subseteq A$.
>
> 1. First, we prove that $A \subseteq B$; that is, $\{6n\colon \text{n} \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.
>
>   Choose any $m \in \{6n\colon n \in \mathbb{Z}\}$. By definition, we can write $m = 6k$ for some $k \in \mathbb{Z}$.

In particular, $m = 2(3k)$, so that $m \in \{3n\colon n \in \mathbb{Z}\}$ as well. By the definition of intersection, we can conclude that $m \in \{2n\colon n \in \mathbb{Z}\} \cap \{3n\colon n \in \mathbb{Z}\}$ as required.

2. Next, we prove that $\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n\colon \text{n} \in \mathbb{Z}\}$.

   Choose any $m \in \{2n\colon n \in \mathbb{Z}\} \cap \{3n\colon n \in \mathbb{Z}\}$, so that both $m \in \{2n\colon n \in \mathbb{Z}\}$ and $m \in \{3n\colon n \in \mathbb{Z}\}$.

   In particular, we can find integers j and k such that $m = 2j$ and $m = 3k$. The fact that $m = 2j$ means that m is even, and so $m = 3k$ is even. But the product of two integers is odd if and only if both intgers are odd; since 4 is odd but $3k$ is even, we conclude that k is even. Then, $k = 2a$ for some integer a, and so $m = 3k = 3(2a) = 6a$, which shows that $m \in \{6n{:}n \in \mathbb{Z}\}$ as required.

Therefore, we can conclude that $A = B$ as we have shown both $A \subseteq B$ and $B \subseteq A$.

c. *Optional:* Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

1. We will show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

   Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. So $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$. In the former case, $(x, y) \in A \times B$, and in the latter case $(x, y) \in A \times C$. So $(x, y) \in (A \times B) \cup (A \times C)$. Since (x, y) was chosen arbitrarily, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

2. We will show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

   Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$. So $x \in A$ and $y \in B \cup C$, and hence $(x, y) \in A \times (B \cup C)$. Since (x, y) was chosen arbitrarily, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Therefore, we can conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Checkpoint 2 — Call over a TA!

Set Algebra

We can also prove that two sets are equivalent using set algebra. More concretely, we can use the stated laws of set algebra to convert one side of the equation to the other (or convert both sides to an identical expression).

Example (from the sample proofs on our website):

Prove that $(A \cap B) \cup (A \setminus B) = A \cap (B \cup (A \setminus B))$.

$$
\begin{aligned}
&(A \cap B) \cup (A \setminus B) && \\
&= (A \cap B) \cup (A \cap \overline{B}) && \text{(Set Difference Law)} \\
&= (A \cap (B \cup \overline{B}) && \text{(Distribution)} \\
&= A \cap U && \text{(Complement Law)} \\
&= A && \text{(Identity Law)} \\
&= A \cap (A \cup B) && \text{(Absorption)} \\
&= A \cap (B \cup A) && \text{(Commutivity)} \\
&= A \cap ((B \cup A) \cap U) && \text{(Identity Law)} \\
&= A \cap ((B \cup A) \cap (B \cup \overline{B}) && \text{(Complement Law)} \\
&= A \cap (B \cup (A \cap \overline{B})) && \text{(Distribution)} \\
&= A \cap (B \cup (A \setminus B)) && \text{(Set Difference Law)}
\end{aligned}
$$

Therefore, the equality holds since all these steps are biconditionally true.

Task 4

a. Let A and B be subsets of some universal set U. Prove that $(A \cap \overline{B}) \cup B = A \cup B$ using set algebra.

$$
\begin{aligned}
&(A \cap \overline{B}) \cup B \\
&= (A \cup B) \cap (\overline{B} \cup B) && \text{(Distributive Law)} \\
&= (A \cup B) \cap U && \text{(Complement Law)} \\
&= A \cup B && \text{(Identity Law)}
\end{aligned}
$$

b. Let A and B be subsets of some universal set U. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ using set algebra.

$$
\begin{aligned}
&(A \setminus B) \cup (B \setminus A) \\
&= (A \cap \overline{B}) \cup (B \cap \overline{A}) && \text{(Set Difference Law)} \\
&= (A \cap \overline{B} \cup B) \cap ((A \cap \overline{B} \cup \overline{A}) && \text{(Distributive Law)} \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) && \text{(Distributive Law)} \\
&= ((A \cup B) \cap U) \cap (U \cap \overline{B} \cup \overline{A}) && \text{(Complement Law)} \\
&= (A \cup B) \cap (\overline{B} \cup \overline{A}) && \text{(Identity Law)} \\
&= (A \cup B) \cap (\overline{A \cap B}) && \text{(DeMorgan's Law)} \\
&= (A \cup B) \setminus (A \cap B) && \text{(Set Difference Law)}
\end{aligned}
$$

c. *Optional:* Show that $(A \setminus B) \setminus (B \setminus C) = (A \cup B) \setminus (A \cap B)$.

$$
\begin{aligned}
&(A \setminus B) \setminus (B \setminus C) \\
&= (A \cap \overline{B}) \cap (\overline{B \cap \overline{B}}) && \text{(Set Difference Law)} \\
&= (A \cap \overline{B}) \cap (\overline{B} \cup C) && \text{(DeMorgan's Law)} \\
&= ((A \cap \overline{B}) \cap \overline{B}) \cup ((A \cap \overline{B}) \cap C) && \text{(Distributive Law)} \\
&= (A \cap (\overline{B} \cap \overline{B})) \cup (A \cap (\overline{B} \cap C)) && \text{(Associative Law)} \\
&= (A \cap \overline{B}) \cup (A \cap (\overline{B} \cap C)) && \text{(Idempotent Law)} \\
&= A \cap (\overline{B}) \cup (\overline{B} \cap C)) && \text{(Distributive Law)} \\
&= A \cap \overline{B} && \text{(Absorption Law)}
\end{aligned}
$$

$$
\begin{aligned}
&= A \setminus B && \text{(Set Difference Law)} \\
&= (A \cap \overline{B} \cup B) \cap ((A \cap \overline{B} \cup \overline{A}) && \text{(Distributive Law)} \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) && \text{(Distributive Law)} \\
&= ((A \cup B) \cap U) \cap (U \cap \overline{B} \cup \overline{A}) && \text{(Complement Law)} \\
&= (A \cup B) \cap (\overline{B} \cup \overline{A}) && \text{(Identity Law)} \\
&= (A \cup B) \cap (\overline{A \cap B}) && \text{(DeMorgan's Law)} \\
&= (A \cup B) \setminus (A \cap B) && \text{(Set Difference Law)}
\end{aligned}
$$

Checkoff — Call over a TA!