Recitation 1

Proofs and Logic

What's recitation?

Recitation is a space for you to work with other members of the CS22 community on problems that we hope will help you hone your understanding of the course material, get better at communicating with other folks about mathematical ideas, and practice for the homework. You'll also get to know some of the TAs and ask them any questions about the course material that you're passionate about, barring specific questions about the homework for the week. If you have any feedback on recitation or the course in general, please share with us through the anonymous feedback form!

Proof Techniques

Direct Proof

A direct proof of a conditional statement $P \Rightarrow Q$ begins by assuming that P is true, and then uses logic, definitions, and standard math facts to deduce that Q is true.

Here is an example:

Example

Claim: If n is odd, then n^2 is odd.

Proof. Suppose n is an odd integer. By definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Then $n^2 = (2k+1)^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2m+1$, such that $m = 2k^2 + 2k$.

Since m is an integer, n^2 is odd. □

a. Prove that the product of an even number and odd number is even.

Solution:

Consider n odd and m even. $n = 2a + 1$ and $m = 2b$.

Their product is $(2a + 1)(2b) = 4ab + 2b = 2(2ab + b)$.

Since $2ab + b$ is an integer their product is even.

b. Prove that the product of two rational numbers is rational.
Hint: A rational number, by definition, can be written in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Solution:

Consider $n = \frac{a}{b}$ and $m = \frac{c}{d}$.

Their product is $nm = \frac{ac}{bd}$. Since the product of two integers is an integer, we have just expressed nm as the ratio of two integers. As b and d are both nonzero, so is their product. Therefore nm is rational.

Negation and Counterexample

Sometimes, we don't ask you to prove claims are true; we instead ask you to prove that they are false! This means your task is to prove the *negation* of the claim.

What is the negation of a claim? Let's think about it.

Question: Suppose Allie says to Joseph: "Everyone at Brown likes dinosaurs!". What would Joseph need to show Allie to convince her this is false? Circle your choice.

(a) Every person at Brown loves dinosaurs.

(b) There is at least one person at Brown who does not like dinosaurs.

(c) There are at least 22 people at Brown who do not like dinosaurs.

Solution:

Answer: b

Question: Write the negation of each proposition below.

a. All CS22 students want to be dinosaur experts.

b. There exists a student in CS22 who is a dinosaur expert.

c. $\forall x \in \mathbb{Z}$, if x is even, $2x$ is odd.

Solution:

1. There exists CS22 student who does not want to be a dinosaur expert.
2. All students in CS22 are not dinosaur experts. **Be sure to note the different between NOT all students are and ALL students are not.**
3. There exists an $x \in \mathbb{Z}$ such that x is even and $2x$ is not odd.

Sometimes, we can show that a claim is false (i.e. the negation of the claim is true) by providing a **counterexample**.

Example

Suppose Joseph makes the claim that, if xy is rational then x and y are rational.

Allie can disprove Joseph's claim by coming up with a counterexample. She notices that Joseph's statement has an implicit "for all" quantifier: "For all x and y, if xy is rational, then x and y are rational." Allie can then negate this statement to get "There exists x and y such that xy is rational and x or y is irrational." If Allie chooses $x = \sqrt{2}$ and $y = \sqrt{2}$, then $xy = 2$ is rational but x and y are irrational. This proves the negated claim, and thus disproves Joseph's original claim. Allie's counterexample was $x = \sqrt{2}$ and $y = \sqrt{2}$.

Question: Now, disprove the following statement by providing a counterexample. If xyz is rational, then x, y, and z are rational.

Solution:

Consider $x = \sqrt{2}$, $y = \sqrt{3}$, $z = \sqrt{6}$. $xyz = 6$ which is rational.

Optional Recommended **Checkpoint — Call over a TA.**

Logic

1. A **proposition** is a statement that evaluates to true or false. For example, "grass is green" is a proposition.

2. A **propositional variable** is a symbol that represents a proposition. Propositional variables are assigned truth values ('true' or 'false'). For example, if we let p represent the proposition "grass is green," then p is a propositional variable.

3. A **propositional formula** can be constructed from atomic propositions via logical connectives. The truth value of a propositional formula can be calculated from the truth values of the atomic propositions it contains.

4. The term **logical expression** is often used synonymously with the word proposition.

5. Two propositions are **logically equivalent** when they have the same truth tables.

6. A proposition is **valid** if it evaluates to true on any choice of inputs; it is true no matter what. That is, a valid proposition is logically equivalent to the expression $(p \vee \neg p)$. This is also called a *tautology*.

7. A proposition is **satisfiable** if it evaluates to true on some choice of inputs. A valid proposition is satisfiable, but so are many propositions which sometimes evaluate to false.

8. If a proposition is **not satisfiable**, it evaluates to false on any choice of inputs; it is false no matter what. That is, it is logically equivalent to the expression $(p \wedge \neg p)$. This is called a *contradiction*.

Truth Tables and Propositions

The truth value of a complex propositional formula is determined by the truth values of its simpler subformulae. For example, the truth value of $(p \Rightarrow r) \wedge q$ is determined by the truth values of $p \Rightarrow r$ and q according to the rules for the conjunction operator $\wedge$. Breaking it down further, the value of $p \Rightarrow r$ is determined by the truth values of p and r according to the implication operator $\Rightarrow$. It follows that the truth value of the whole propositional formula $(p \Rightarrow r) \wedge q$ is determined by the truth values of p, q, r according to the rules for $\wedge$ and $\Rightarrow$.

To develop a systematic way of checking the truth values of a propositional formula under each assignment of truth values to its constituent propositional variables, we use truth tables.

The **truth table** of a propositional formula is the table with one row for each possible assignment of truth values to its constituent propositional variables, and one column for each subformula (including the propositional variables and the propositional formula itself).

Let's now review a truth table corresponding to some important logical connectives:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Note: $p \oplus q$ represents XOR (exclusive or).

Logical Equivalence: Two Approaches

You have two techniques at your disposal to determine if two expressions are logically equivalent: by using truth tables or by using logical rewrite rules.

Given two expressions, you can write out the truth table for each one. If they have the same inputs and their truth tables are the same, they are logically equivalent.

Or, given two expressions, you use logical equivalence rules to try to get from one expression to the other. For example, take the expression $\neg(x \wedge y)$. Using DeMorgan's law, you could get that the not "distributes" in this expression, and it is therefore equivalent to $(\neg x \vee \neg y)$. A full list of important logical equivalences (which will be very useful going forward) can be found on the course website.

Task: Determine if each pair of expressions below are logically equivalent (i.e. determine if the given equivalence holds). You can choose to either use truth tables or simplify the expressions to arrive at your answer. To show that the expressions are not logically equivalent, you only need to provide one counterexample.

a. $p \vee q \equiv (\neg p) \wedge (\neg q)$

Solution:

False. A counterexample is when p and q are both True. The full truth table is

p	q	$p \vee q$	(NOT p) AND (NOT q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

b. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Solution:

True. The full truth table is

p	q	p IFF q	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

c. *Optional:* $p \wedge \neg q \equiv \neg(\neg p \vee q)$

Solution:

True. The full truth table is

p	q	$p \wedge \neg q$	NOT$(\neg p \vee q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

d. *Optional:* $((p \vee q) \wedge z) \equiv (p \vee (q \wedge z))$

Solution:

False. A counterexample is when p is True, q is True, and z is False. The full truth table is

p	q	z	$((p \lor q) \land z)$	$(p \lor (q \land z))$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

More Practice

a. Give an assignment to the variables x_1, x_2, x_3 which makes the following logical expression evaluate to true.

$(x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land x_3$

Solution:

x_1 is true, x_2 is false, x_3 is true

b. Is the following logical expression valid? Explain your answer.

$(x_1 \land x_2) \lor (\neg x_1 \land \neg x_2 \land \neg x_3) \lor (x_3 \land \neg x_3)$

Solution:

Nope, consider x_1 is false and x_2 is true.

c. Come up with a logical expression with three variables which has **only one** assignment to the variables which makes it true.

Solution:

All the variables anded together

Facebook Censorship Rules

In 2017, Propublica allegedly released information on how Facebook determines what constitutes hate speech. The ruleset they outlined was a highly deterministic, logic-based system (note the difference between logic-based and logical).

Here are several overarching censorship rules that Facebook follows to determine if a comment is hate speech (note that enforcement is inconsistent at best, and it is unknown whether or not Facebook has changed its methodology):

1. A comment in question is making an "attack" or hateful comment (the specifics on how this is determined is not clear).

2. The comment in question is directed towards **people** belonging to a "protected group" or any combination of protected groups. Note that each member of a protected group is protected indiscriminately (comments disparaging "man" are weighed equally as comments disparaging "woman") and the comment must be directed towards people, not ideologies.

   - The protected groups are race, sex, gender identity, religious affiliation, national origin, ethnicity, sexual orientation, and serious disability/disease. This group can shift depending on current events; for example, "Syrian immigrants" were added as a protected group in the wake of the Syrian refugee crisis.

3. If the target of the comment has multiple listed characteristics, (e.g. "Asian" has one characteristic, "Asian male" has two, and "elderly Asian male" has three), **each characteristic must be in a protected group** for the comment to be considered hate speech.

If these conditions are met, Facebook flags the comment as hate speech and the comment is subject to being taken down. We can express this as a proposition: let p be that the comment is hateful, let q_i be that the ith characteristic of the targeted group belongs in the "protected groups," and let r be that the given comment is hate speech. We apply this simple logic to determine whether the statement is hate:

$$p \wedge q_1 \wedge q_2 \wedge q_3 \cdots \to r$$

Task: Given the following list of subgroup characteristics, determine if the following groups would be protected from "hate speech" under the Facebook rules (focus on rules three and two). Give another example of a group that should be protected but isn't.

1. Female driver
2. Black kid
3. White man
4. Islam
5. Muslims

Solution:

1. "Female" is protected under sex; "driver" is not protected under occupation. "Female driver" is not protected.
2. "Black" is protected under race; "kid" is not protected under age. "Black kid" is not protected.
3. "White" is protected under race; "man" is protected under gender. "White man" is protected.
4. "Islam" is not protected because comments are directed at the religion, instead of at people.
5. "Muslims" is protected under religion.

There are many other groups that should be protected but aren't. Students must generate a group that contains at least one descriptor not found in the "protected groups" list (race, sex, gender identity, religious affiliation, national origin, ethnicity, sexual orientation and serious disability/disease). Remember that Facebook uses colorblind rules, so "white" is just as protected as "black."

Task: In 2018, Facebook responded to the article by adding "age" into the list of protected groups. How have your answers to the previous tasks changed? How could someone continue to post racist, sexist, and hateful comments with these rules? Does something still strike you wrong about the list of protected groups and unprotected groups?

Solution:

Pass condition: Students must be able to identify that "female driver" and "Islam" remain unprotected despite the rule change. Students must be able to describe a way in which users can continue attacking protected groups by adding descriptors to the subject (e.g. using "arguments" against trans people in the military as a vessel to be transphobic) or attacking ideologies rather than people (e.g. attacking "feminism" as a vessel for sexist comments). Students should be able to show that not all groups that warrant protection will continue to have protection and that Facebook's colorblind model protects those who experience discrimination just as much as people who don't (e.g. black is just as protected as white).

Prodding Questions:

- How might Facebook rules be different regarding hate speech towards people vs. hate speech towards an institution?
- Do you believe that the revised set of protected groups is sufficient protection? Would Facebook be able to implement tighter restrictions while still protecting free speech?

Task: Facebook tried its best to use a rigid set of predicates and propositions to determine whether a given comment is hate speech. Describe one limitation that logical formulas have in terms of applying them to complicated real-world situations.

Solution:

Pass condition: Students should be able to identify how the real world relies on imperfect information and assumptions. Logic relies on knowing, for a fact, that something is true or false, yet the real world rarely has that sort of determinism. Formulas are also limited in scope, so not every contributing variable can be accounted for (e.g. current events and the demographic landscape of the country the comment originated from).

Examples:

- Logical formulas do not account for all possible outcomes in the real world; while there are infinite possibilities and factors that can influence outcomes in the real world, logical formulas can only account for a finite number of them, leading to room for error.
- Logical formulas and algorithms follow a set of rules and patterns; human behavior does not. Algorithms are also created by humans, and as such can already have built-in biased behavior. These algorithms change and evolve based on observations of human behavior, and as we have already seen, humans

have a long history of bias and questionable moral behavior.

Prodding Questions:

- Look back at the proposition we wrote out that describes Facebook's behaviors. Is it always easy to say when p is true?
- If we can't accurately predict when a given variable is true or not, can we be sure that the logical formula has a logical conclusion?
- Can real-world situations always be reduced down to a series of true-false questions, or are there times when neither option is entirely correct?
- Imagine you had to explain what "hate" is only using true-false statements. Could you get all the nuance the situation has with just yes-or-no questions?

Proof by Contradiction

In math, a proposition and its negation cannot both be true. This would be a **contradiction**. For example, it cannot be the case that 3 is odd and 3 is not odd. Only one of these statements can be true, and one of them has to be true.

This idea motivates a proof technique called *proof by contradiction*. The general idea is this:

Say we are trying to prove a proposition T.

To prove T is true by contradiction:

1. Begin by assuming T is NOT true. That is, we assume that the negation of T is true.
2. Assuming T is not true leads us to a contradiction. That is, by making logical leaps from T being not true, we arrive at the fact that a statement x and its negation both have to be true.
3. But x and its negation cannot both be true; this is a contradiction. We got to this contradiction by assuming T was false. Therefore, we know T cannot be false; i.e. T is true.

Here is an example.

Example

Claim: There is no smallest rational number greater than 0.

Proof. Assume for sake of contradiction that there exists a smallest rational number, say r.

However, $r/2$ is a rational number greater than 0 and smaller than r.

This is a contradiction to the fact that r is the smallest rational number.

Assuming that there is a smallest rational number led to a contradiction, and therefore there is no smallest rational number greater than 0. □

Now it's your turn! Prove the following statements by contradiction.

a. 2 is an even number.

Solution:

Assume for sake of contradiction that 2 was odd. Then $2 = 2k + 1$ for some integer k. Then $k = \frac{1}{2}$ which is a contradiction since k is an integer.

b. Suppose $a, b \in R$. If a is rational and ab is irrational, then b is irrational.

Solution:

Since a is rational, $a = \frac{x}{y}$ for some $x, y \in \mathbb{Z}, y \neq 0$. Assume for sake of contradiction that b is rational. Then $b = \frac{z}{w}$ for some $z, w \in \mathbb{Z}, w \neq 0$. This means $ab = \frac{xz}{yw}$ and $xz, yw \in \mathbb{Z}, yw \neq 0$, so ab is rational, which is a contradiction.

Proof by Contrapositive

If we have a statement "if A then B", we define the contrapositive of that statement to be "if not B then not A". If the statement is true, then the contrapositive is true, and if the statement is false, then the contrapositive is false (check this for yourself using truth tables!). This logical equivalence brings about a powerful proof technique: *proof by contraposition.* We can prove a statement just by showing that its contrapositive holds true.

Say we are trying to prove a proposition in the form $p \Rightarrow q$. To prove by contraposition:

- Assume $\neg q$ is true. In other words, assume the negation of q is true.
- Show $\neg p$ is true. In other words, show that p is false.

Note: The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. The steps above are really just proving the contrapositive.

Example

Let's say we want to prove the statement *if x^2 is even, then x is even.* We could spend the effort to come up with a new direct proof of this. But, earlier in the recitation (the direct proof example) we already showed that *Suppose n is odd. Then, n^2 is odd.* This is exactly the contrapositive of what we want to show, so we can just reuse that proof for this new statement:

Proof. We will prove the contrapositive: *if x is odd, then x^2 is odd.*

Suppose x is an odd integer. By definition, $x = 2k + 1$ for some $k \in \mathbb{Z}$.

Then $x^2 = (2k+1)^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2m+1$, such that $m = 2k^2 + 2k$.

Since m is an integer, x^2 is odd.

Thus, we have proven via contraposition that *if x^2 is even, then x is even.* □

Task 1

Write the contrapositive for each of these statements.

1. If a student attends recitation, then they understand the material.
2. If someone likes all dinosaur species, then they like chickens.
3. A student is a Gemini if they are born on May 21.

Solution:

1. If a student does not understand the material, then they did not attend recitation. (NOT: If a student does not attend recitation, then they do not understand the material)
2. If someone does not like chickens, then they do not like all dinosaur species.
3. If a student is not a Gemini, then they are not born on May 21. (NOT: If a student is not born on May 21, then the student is not a Gemini.)

Checkpoint — Call a TA over.

Checkoff - call over a TA!

If you weren't able to complete recitation, you may come to *any* recitation in the next week to get it checked off (optimally with your group). We *highly recommend* you to do this at the same time next week, with your group, *after* you have finished next week's recitation. The deadline for recitation checkoff is by your next scheduled recitation time.