Homework 8

Due: ***November 17, 2025***

All homeworks are due at 6:00 PM on Gradescope. You will receive full credit for this homework for turning anything in on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Problem 1

Suppose you roll three 6-sided die in order and let x_i be the value of the ith die that you roll. For instance, you might roll 5 then 1 then 5 in which case $x_1 = 5$, $x_2 = 1$ and $x_3 = 5$. We refer to (x_1, x_2, x_3) as the sequence rolled.

a. What is the probability that the rolled sequence is $(1, 1, 1)$?

b. What is the probability that exactly 2 of the 3 rolled values are 6?

c. We say that a sequence is *non-decreasing* if $x_i \leq x_j$ for all i, j such that $i < j$. For instance the sequence $(1, 1, 5)$ is non-decreasing but the sequence $(1, 5, 2)$ is not (since $5 < 2$). What is the probability that the rolled sequence is non-decreasing.

Solution:

a. Since the relevant probability space is just the uniform space over sample space $S = \{1, 2, 3, 4, 5, 6\}^3$, the probability of a fixed outcome is just $1/|S|$. By the product rule we have $|S| = 6^3 = 216$ and so the probability is $1/216$.

b. Since, as in the previous part, the relevant probability space is just the uniform space over sample space $S = \{1, 2, 3, 4, 5, 6\}^3$, it suffices to count the number of sequences in $\{1, 2, 3, 4, 5, 6\}^3$ that have exactly two 6s. We can break down the process of producing such a sequence by choosing 2 indices in $\{1, 2, 3\}$ and then assigning a number in $1, 2, 3, 4, 5$ to the remaining index. By the product rule, the number of such ways of doing this is

$$\binom{3}{2} \cdot 5 = 15$$

and so our final probability is $\frac{15}{|S|} = \frac{15}{216}$.

c. As in the previous part, it suffices to count the number of non-decreasing sequences. Observe that by the sum and product rule, the number of non-decreasing sequences consisting of 2 numbers in $\{1, 2, \ldots, n\}$ is $n + (n-1) + (n-2) + \ldots + 1 = \sum_{i=1}^{n} i$. Then, again, by the sum and product rule we have that the number of non-decreasing sequences of length 3 in $\{1, 2, 3, 4, 5, 6\}^3$ is

$$\sum_{j=1}^{6} \sum_{i=1}^{j} i$$

which is

$$(1) + (1+2) + (1+2+3) + (1+2+3+4) + \\ (1+2+3+4+5) + (1+2+3+4+5+6)$$

which is 56 so our final probability is 56/216.

Problem 2

Suppose a grocery store sells apples and bananas. If 40% of customers buy apples, 35% buy bananas, and 15% buy both, what is the probability that a uniformly random customer:

1. Doesn't buy apples?
2. Buys apples but not bananas?
3. Buys apples or bananas?

Solution:

1. By the complement rule we have that it is .6.
2. By the set difference rule it is $.4 - .15 = .25$.
3. By the subtraction rule it is $.4 + .35 - .15 = .6$.

Problem 3

In this problem we'll derive a lower bound on the probability of the intersection of events.

a. Let $(S, \Pr)$ be a discrete probability space and let $E, F \subseteq S$ be events (not necessarily independent). Prove that $\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1$.

HINT: There are multiple ways to prove this. In one, you should consider the fact that $E \cup F$ and $\overline{E \cup F}$ are disjoint and their union is all of S.

b. More generally, let $E_0, E_1, \ldots, E_n \subseteq S$ be events. Use induction to prove that $\Pr(E_0 \cap E_1 \cap \ldots \cap E_n) \geq \Pr(E_0) + \Pr(E_1) + \ldots + \Pr(E_n) - n$.

Solution:

a. We can decompose the sample space S into two disjoint events, $E \cup F$ and $\overline{E \cup F}$.

By the sum rule, $\Pr(E \cup F) + \Pr(\overline{E \cup F}) = 1$.

By the subtraction rule, $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$.

Combining these two equations, we get that

$$1 = \Pr(E) + \Pr(F) - \Pr(E \cap F) + \Pr(\overline{E \cup F})$$

or equivalently

$$\Pr(E \cap F) = \Pr(E) + \Pr(F) - 1 + \Pr(\overline{E \cup F}).$$

Since $\Pr(\overline{E \cup F}) \geq 0$, this implies that $\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1$ as desired.

b. We proceed by basic induction with the predicate $P(n) = \Pr(E_0 \cap \ldots \cap E_n) \geq \Pr(E_0) + \ldots + \Pr(E_n) - n$.

Base case: we want to show $P(0)$, that is, $\Pr(E_0) \geq \Pr(E_0) - 0$. This follows since $\Pr(E_0) = \Pr(E_0) - 0$.

Inductive step: Fix an arbitrary natural number n and suppose $P(n)$, that is, that $\Pr(E_0 \cap \ldots \cap E_n) \geq \Pr(E_0) + \ldots + \Pr(E_n) - n$. We want to show $P(n+1)$, that is, $\Pr(E_0 \cap \ldots \cap E_n \cap E_{n+1}) \geq \Pr(E_0) + \ldots + \Pr(E_n) + \Pr(E_{n+1}) - (n+1)$.

For convenience, let A denote $E_0 \cap \ldots \cap E_n$. From the first part we know that $\Pr(A \cap E_{n+1}) \geq \Pr(A) + \Pr(E_{n+1}) - 1$. And from our induction hypothesis, $\Pr(A) \geq \Pr(E_0) + \ldots + \Pr(E_n) - n$. So we compute

$$\Pr(E_0 \cap \ldots \cap E_n \cap E_{n+1}) = \Pr(A \cap E_{n+1})$$

$$\begin{aligned}
&\geq \Pr(A) + \Pr(E_{n+1}) - 1 \\
&\geq (\Pr(E_0) + \ldots + \Pr(E_n) - n) + \Pr(E_{n+1}) - 1 \\
&= \Pr(E_0) + \ldots + \Pr(E_n) + \Pr(E_{n+1}) - (n+1)
\end{aligned}$$

as desired.

Problem 4

A box has 3 red and 1 green ball. You draw one ball uniformly random. If the ball you draw is red, you put it back and add one more red ball. If the ball you draw is green, you put it back and add one more green ball. You then draw a second ball uniformly at random.

What is the probability that the second ball is green?

Solution:

Let G_i and R_i be the event of the ith ball being green and red respectively. We want to compute $\Pr(G_2)$. Since R_1 and G_1 partition the sample space, by the law of total probability we have

$$\Pr(G_2) = \Pr(G_2 \mid G_1) \cdot \Pr(G_1) + \Pr(G_2 \mid R_1) \cdot \Pr(R_1) \tag{1}$$

Similarly, by the starting conditions we have

$$\Pr(G_1) = \frac{1}{4}$$

and

$$\Pr(R_1) = \frac{3}{4}.$$

Likewise

$$\Pr(G_2 \mid G_1) = \frac{2}{5}$$

and

$$\Pr(G_2 \mid R_1) = \frac{1}{5}.$$

Plugging this all back in to Equation (1) we get

$$\Pr(G_2) = \frac{2}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{4} = \frac{1}{4}.$$

Problem 5

Prove that given any discrete probability space $(S, \Pr)$ and events $A, B \subseteq S$, if A is independent of B then B is independent of A.

Solution:

Fix an arbitrary $(S, \Pr)$ and events $A, B \subseteq S$. Suppose A is independent of B so

$$\Pr(A \mid B) = \Pr(A).$$

Applying the definition of conditional probability, the fact that $A \cap B = B \cap A$ and $\Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B)$, we get

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A \mid B) \cdot \Pr(B)}{\Pr(A)} = \frac{\Pr(A) \cdot \Pr(B)}{\Pr(A)} = \Pr(B)$$

as required to show B is independent of A.

Problem 6

Suppose you flip two fair coins. Let X be the number of heads you get.

1. What is $\mathbb{E}[X]$?
2. What is $(\mathbb{E}[X])^2$? Figuring this out should be easy if you know $\mathbb{E}[X]$.
3. What is $\mathbb{E}[X^2]$? Figuring this out should require some more thought even though you already know $\mathbb{E}[X]$.

Solution:

1. It is $\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 = 1$.
2. It is $1^2 = 1$.
3. It is $\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1^2 + \frac{1}{4} \cdot 1^2 + \frac{1}{4} \cdot 2^2 = 1.5$.

Problem 7

You have two magical dice. The first die is a fair six-sided die, with each face numbered 1 through 6 and equally likely to appear. Let X be the random variable giving the outcome of the first die roll.

The result of the first die roll determines the weighting of the second die. Let Y give the outcome of the second die roll. Then, for $k \in \{1, 2, 3, 4, 5, 6\}$, if $X = k$ then $\Pr(Y = 1) = \frac{k}{6}$ and $\Pr(Y = i) = \frac{1}{5} - \frac{k}{30}$ for $i \in \{2, 3, 4, 5, 6\}$.

For example, if the first die rolls a 2, then the second die has a $\frac{1}{3}$ of landing on 1 and a $\frac{2}{15}$ chance of landing on each of $2, 3, 4, 5, 6$.

1. What is the probability of $Y = 1$?
2. What is the probability of $Y = 3$?

Solution:

a. By definition we have

$$\Pr(Y = 1 \mid X = x) = \frac{x}{6}.$$

The first die is fair, so $P(X = x) = \frac{1}{6}$ for any $x \in \{1, 2, 3, 4, 5, 6\}$. Thus by the law of total probability we get

$$\Pr(Y = 1) = \sum_{x=1}^{6} \Pr(X = x) \cdot \Pr(Y = 1 \mid X = x).$$

Substituting the probabilities we get

$$\Pr(Y = 1) = \sum_{x=1}^{6} \frac{1}{6} \cdot \frac{x}{6}.$$

Simplifying we get

$$\Pr(Y = 1) = \frac{1}{36} \sum_{x=1}^{6} x = \frac{1}{36} \cdot 21 = \frac{21}{36}.$$

b. For $Y = 3$ we have:

$$\Pr(Y = 3 \mid X = x) = \frac{6 - x}{30}$$

By the law of total probability, we get

$$\Pr(Y = 3) = \sum_{x=1}^{6} P(X = x) \cdot P(Y = 3 \mid X = x).$$

Since $P(X = x) = \frac{1}{6}$ for any $x \in \{1, 2, 3, 4, 5, 6\}$, we get

$$\Pr(Y = 3) = \sum_{x=1}^{6} \frac{1}{6} \cdot \frac{6 - x}{30}.$$

Simplifying, we get

$$\Pr(Y = 3) = \frac{5}{180} + \frac{4}{180} + \frac{3}{180} + \frac{2}{180} + \frac{1}{180} + \frac{0}{180}$$

Thus,

$$\Pr(Y = 3) = \frac{15}{180} = \frac{1}{12}$$

Problem 8 (Mind Bender)

Let $S = \{1, 2, 3, \ldots, 9\}$ be the numbers 1 through 9 and let π be a uniformly random permutation of S where π_i is the ith element of π. For instance, we could have $\pi = (1, 2, 5, 6, 3, 8, 9, 4, 7)$ in which case $\pi_3 = 5$. In a slight abuse of notation, we also define π_{10} as π_1.

Next, define the random variable $X(\pi) = \sum_{i=1}^{9} |\pi_{i+1} - \pi_i|$. For instance, if we have that $\pi = (1, 2, 5, 6, 3, 8, 9, 4, 7)$ then we would have that

$$\begin{aligned} X(\pi) &= |2-1| + |5-2| + |6-5| + |3-6| + |8-3| + |9-8| + |4-9| + |7-4| + |1-7| \\ &= 1 + 3 + 1 + 3 + 5 + 1 + 5 + 3 + 6 \\ &= 28. \end{aligned}$$

What is the probability that X achieves its minimum possible value? In particular if π_0 is the permutation which minimizes X, then what is $\Pr(X = X(\pi_0))$?

HINT: The minimum of the sum is 16, and it is minimized by any cyclic ordering, but not only cyclic orderings! Start by proving that the sum is minimized by any permutation π such that there is an index i where $\pi_1 \ldots \pi_i$ is increasing (i.e. if $j < k \leq i$ then $\pi_j < \pi_k$), and $\pi_i \ldots \pi_9$ is decreasing (i.e. if $j < k \leq i$ then $\pi_j > \pi_k$).

Solution:

Problem and solution courtesy of (Brown alum!) Roie Levin.

This problem is trickier than it looks. The minimum of the sum is 16, and it is minimized by any cyclic ordering, but not only cyclic orderings! It is actually minimized by any permutation π such that $\pi_1, \ldots, \pi_9$ for which there is an index i such that $\pi_1 \ldots \pi_i$ is increasing, and $\pi_i \ldots \pi_9$ is decreasing.

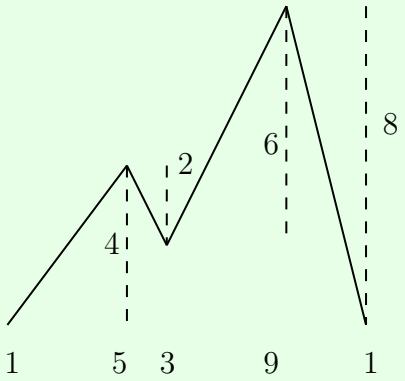
The reason for this is that if $\pi_j \leq \pi_{j+1} \leq \pi_{j+2}$, then

$$|\pi_{j+1} - \pi_j| + |\pi_{j+2} - \pi_{j+1}| = |\pi_{j+2} - \pi_j|.$$

Hence if we can break our sequence into monotone (i.e. increasing or decreasing) subsequences $\pi_1, \ldots, \pi_{i_1}$ and $\pi_{i_1}, \ldots, \pi_{i_2}$ and $\pi_{i_2}, \ldots \pi_{i_3}$ etc., then we can also express the sum as

$$\sum_i |\pi_{i+1} - \pi_i| = \sum_k |\pi_{i_{k+1}} - \pi_{i_k}|.$$

For example, if our ordering is $1, 2, 4, 5, 3, 6, 7, 9, 8$, we could break this into $1 \ldots 5$, $5 \ldots 3$, $3 \ldots 9$, $9 \ldots 1$ would pay $4 + 2 + 6 + 8$. Visually:



The point is that these subsequences define intervals that have to span $[1, 9]$ twice, once up, once down, and we pay for the sum of the lengths of the intervals: the cheapest way is to have two subsequences $\{1, \ldots, 9\}$ and $\{9, \ldots 1\}$. (We could make this formal by saying each unit interval makes us pay 1 each time it is covered by a sequence, and it will be covered at least twice, so exactly twice is best possible.)

To understand the probability of such an ordering, let us pretend by cyclic shifting that $\pi_1 = 1$ (the other cases are symmetric and equally likely). There are 8! total permutations that fix 1 this way. The number of permutations that fix 1 this way *and* have the increasing/decreasing property is the number of permutations for which all the numbers from $\{2, \ldots, 8\}$ to the left of 9 are in increasing order, and all the numbers from $\{2, \ldots, 8\}$ to the right are in decreasing order: i.e. there is exactly one such ordering for every partition of $\{2, \ldots, 8\}$ into left/right sets, so 2^7. Hence the total probability is $2^7/8! = 1/315$.