Homework 7

Due: ***November 3, 2025***

All homeworks are due at 6:00 PM on Gradescope. You will receive full credit for this homework for turning anything in on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Problem 1

Show that all of the following sets have cardinality equal to the cardinality of the integers $|\mathbb{Z}|$ by giving a function that is a bijection from the set to $\mathbb{Z}$; you don't need to prove that your function is bijective, just give the function.

a. The even integers $\mathbb{Z}_{\text{even}}$.

b. The natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

c. All triples of natural numbers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Solution:

a. Define as our bijection $f(n) = \frac{n}{2}$.

b. Define as our bijection

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

c. Let g be the bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ that we saw in class and let h be the bijection from $\mathbb{N}$ to $\mathbb{Z}$ from the previous part. Then, given a triple of natural numbers (a, b, c) for $a, b, c \in \mathbb{N}$, we let our bijection be

$$f((a, b, c)) = h(g(g(a, b), c)).$$

Problem 2

Ellis is choosing a new exciting password. The requirements for his (case-sensitive) password are that:

- The password must consist of 5 or 6 valid characters.
- A valid character is either a lower case letter ("a" through "z" of which there are 26), an upper case letter ("A" through "Z" of which there are 26) or the symbol "!".
- The password must include exactly one "!".

For example, some valid and distinct passwords are "CSYa!", "CsYa!" and "Cs!Yay".

a. Give the number of possible passwords. Justify your answer using (formal or informal) counting rules.

b. Suppose we add the additional constraint to the passwords that

   - The first letter can be upper or lower case (even if it occurs after the "!") but each letter must be the opposite case of the previous letter. Here, if a letter occurs just after "!" but is not the first letter, then it must be the opposite case of the letter just before the "!"

   For example, with this additional constraint, some valid and distinct passwords are "CsYa!", "cSyA!", "Cs!YaY", "cS!yAy", "!cSyAy" and "!CsYaY".

   Give the number of possible passwords with this additional constraint. Justify your answer using (formal or informal) counting rules.

Solution:

a. We can break the task of choosing an n-character password for $n \in \{5, 6\}$ into choosing the position for our required "!" and then, for each of our remaining $n - 1$ characters, choosing a lower or upper case letter. The number of choices for our "!" is n. By the sum rule, the number of choices of lower or upper case letters is $26 + 26 = 52$.

   Thus, by the product rule, the number of n-character passwords for $n \in \{5, 6\}$ is therefore

$$n \cdot 52^{n-1}.$$

   Lastly, by the sum rule the number of possible passwords is equal to the number

of 5-character passwords plus the number of 6-character passwords and so the total number of passwords is

$$6 \cdot 52^5 + 5 \cdot 52^4 = 2317782272.$$

b. We can break the task of our choosing an n-character password for $n \in \{5, 6\}$ into choosing the position for our required "!", choosing the case for the first letter and then choosing each letter (without regards to their case). Again, the number of choices for our "!" is n. There are 2 possible choices for the case of the first letter. Once we choose the case of the first letter, this fixes the case of every letter and so there are only 26 possible choices for every letter (including the first letter).

Thus, by the product rule, the number of n-character passwords for $n \in \{5, 6\}$ is therefore

$$n \cdot 2 \cdot 26^{n-1}.$$

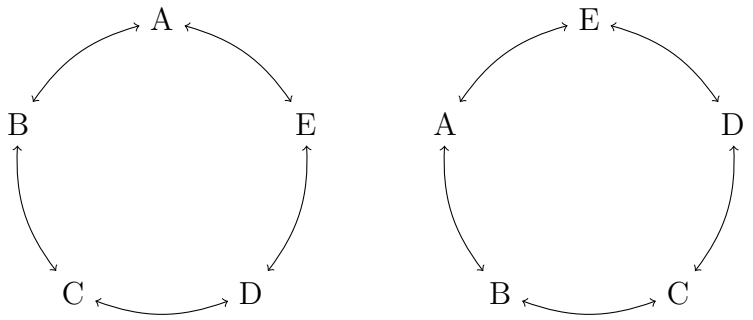
Lastly, by the sum rule the number of possible passwords is equal to the number of 5-character passwords plus the number of 6-character passwords and so the total number of passwords is

$$6 \cdot 2 \cdot 26^5 + 5 \cdot 2 \cdot 26^4 = 147146272.$$

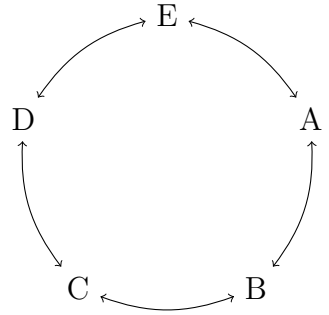
Problem 3

Five people, labeled **A**, **B**, **C**, **D**, and **E**, are standing in a circular standoff. Two standoffs are considered the same if they involve the same participants in the same clockwise order.

For instance, these standoffs are the same:



This one is different from the two above:



a. Unbeknownst to the others, **A** and **B** have secretly made a pact and must always stand next to each other. In how many ways can the five people form their standoff? Justify your answer.

b. Two more people, **F** and **G**, hear about the standoff and want to join, but there is only room for 5 people in the circle. In how many ways can the seven people form their standoff? (Assume **A** and **B** can now stand anywhere.) Justify your answer.

c. Suppose **A** and **B** must stand next to each other, but only if they are both among the 5 chosen for the standoff. (For example, if **B** is in the standoff and **A** is not, then there are no restrictions on where **B** stands.) In how many ways can the seven people form their standoff under this condition? Justify your answer.

Solution:

a. If **A** and **B** must always stand next to each other, treat them as one unit. Then we have 4 entities arranged in a circle, so $(4-1)! = 3! = 6$ arrangements by the generalized product rule. Since **A** and **B** can switch order, there are $2 \cdot 6 = 12$ possible standoffs.

b. There are $\binom{7}{5} = 21$ ways to choose which 5 of the 7 people stand in the circle, and $(5-1)! = 4! = 24$ ways to arrange them around the circle. Hence, there are $21 \cdot 24 = 504$ possible standoffs by the generalized product rule.

c. We divide into four cases based on which of **A** and **B** are in the standoff, applying the generalized product and division rule in each one.

   (a) Suppose **A** and **B** are both in the standoff. We must then choose 3 more people for which there are $\binom{5}{3} = 10$ ways of doing so. From part (a), there are 12 ways to arrange them so there are $10 \cdot 12 = 120$.

   (b) Suppose **A** and **B** are both not in the standoff. Then the remaining 5 people must be in the circle and can be arranged in any standoff for $(5-1)! = 24$ arrangements.

   (c) Suppose **A** is in but **B** is not. We must choose 4 of the 5 remaining other people which can be done in $\binom{5}{4} = 5$ and then these 5 can be arranged in any way for 24 possibilities and so $5 \cdot 24 = 120$ total.

   (d) Suppose **B** is in but **A** is not. This is the same as in the previous case where we have switched the roles of **B** and **A** so it gives another 120 standoffs.

   By the sum rule, we get $120 + 24 + 120 + 120 = 384$ total standoffs.

Problem 4

Consider the following equation with variables $\{x_1, x_2, x_3\}$:

$$x_1 + x_2 + x_3 = 200$$

In each of the following parts, be sure to justify your answer.

a. Count the number of solutions to this equation which satisfy $x_i \geq 0$ and $x_i \in \mathbb{Z}$ for every i. Some example distinct solutions are: $x_1 = 0$, $x_2 = 190$, $x_3 = 10$, and $x_1 = 198$, $x_2 = 1$, $x_3 = 1$.

b. Now, suppose we additionally require that our solution has $x_1 > 0$. Count the number of solutions with this additional constraint.

Solution:

a. This is an example of "stars and bars." In particular, we need two "bars" to split 200 "stars" between x_1, x_2, and x_3. The formula for stars and bars says that the number of such placements is

$$\binom{202}{2} = 20301.$$

b. Let $u_1 = x_1 - 1$. Notice that $x_1 > 0$ iff $u_1 \geq 0$. Thus, we may substitute $x_1 = u_1 + 1$ to get that our goal is a solution to

$$u_1 + 1 + x_2 + x_3 = 200$$

where $u_1, x_2, x_3 \geq 0$ which is equivalent to

$$u_1 + x_2 + x_3 = 199$$

where $u_1, x_2, x_3 \geq 0$.

As in the previous part, this is a stars and bars counting problem with 2 bars and 199 stars and so there are

$$\binom{201}{2} = 20100.$$

Problem 5

Suppose that n is a positive integer divisible by 12. Let $A = \{4, 8, 12, \ldots, n\}$ be all multiples of 4 up to n and let $B = \{6, 12, 18, \ldots, n\}$ be all multiples of 6 up to n. Show that $|A \cup B| = \frac{n}{3}$.

Solution:

By the subtraction rule we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We determine each of these cardinalities separately.

$|A|$ is $\frac{n}{4}$ because n is divisible by 12 and therefore divisible by 4 meaning that every fourth number of $\{1, 2, \ldots, n\}$ is in A.

$|B|$ is $\frac{n}{6}$ because n is divisible by 12 and therefore divisible by 6 meaning that every sixth number of $\{1, 2, \ldots, n\}$ is in B.

$A \cap B$ is $\{12, 24, \ldots\}$—in other words, the multiples of 12. As such, since n is divisible by 12 we have that every 12th number is in $A \cap B$ so $|A \cap B| = \frac{n}{12}$.

Thus, we get

$$|A \cup B| = |A| + |B| - |A \cap B| = \frac{n}{4} + \frac{n}{6} - \frac{n}{12} = \frac{4n}{12} = \frac{n}{3}.$$

Problem 6

Prove that if A is a set of $n+1$ integers, then there are two distinct $a, b \in A$ such that $n \mid a - b$.

Solution:

Let $r_1, r_2, \ldots, r_{n+1}$ be the remainders of these numbers divided by n. Since these are in $[0, n)$, there are only n possible values for the remainders. Thus, by the pigeonhole principle there are 2 numbers, a and b among our $n+1$ numbers such that

$$a \equiv b \mod n.$$

However, since $a \equiv b \mod n$ iff $n \mid a - b$, the theorem follows.

Problem 7 (Mind Bender)

You have 101 dalmatians where the ith dalmatian has s_i spots for some non-negative $s_i \in \mathbb{Z}$. Prove that there is a non-empty subset of your dalmatians whose total number of spots is divisible by 101.

Solution:

For any $k \in [1, 101]$ Let $S_k = s_1 + s_2 + \ldots + s_k$. Notice that there are 101 possible S_ks. Let $r_i = \text{rem}(S_k, 101)$ be the remainder of S_k when dividing by 101. Since such a remainder is in $[0, 101)$, there are only 101 possible values for it.

If for some S_k we have $S_k \equiv 0 \mod 101$, it follows that the first k dalmatians are such that the sum of their spot numbers is congruent to 0 mod 101 and so is divisible by 101.

Otherwise, there is no S_k such that $S_k \equiv 0 \mod 101$. Treating the possible remainders in $(0, 101)$ as holes, we get by the pigeonhole principle that there must be a k and $k' > k$ such that $r_k = r_{k'}$. It follows that $S_k \equiv S_{k'} \mod 101$ and so $S_{k'} - S_k \equiv 0 \mod 101$. However, $S_{k'} - S_k$ is exactly the total number of the spots of the dalmatians numbered $k+1, k+2, \ldots, k'$. Thus, this set of dalmatians has a total number of spots which is $\equiv 0 \mod 101$ and therefore divisible by 101.