Homework 7

Due: Friday, April 5, 2024

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

For each of the statements below, determine if it is *always* true, *sometimes* true, or *never* true. Justify your answers. To justify an "always" or "never" answer, write a proof; to justify a "sometimes" answer, give one witness that makes the statement true and one that makes the statement false, explaining these judgments.

For example, the statement

> Let $a, b : \mathbb{N}$ and suppose $a \mid b$. Then the greatest prime factor of b divides a.

is *sometimes* true. It is true if $a = 6$ and $b = 12$, since $6 \mid 12$ and the greatest prime factor of 12 is 3, which divides 6. It is false if $a = 2$ and $b = 6$, since $2 \mid 6$ but the greatest prime factor of 6 is 3, which does not divide 2.

a. Let $p, q, r, s : \mathbb{N}$ be prime numbers and suppose that $pq = rs$. Then $p = r$ and $q = s$.

b. Let $p : \mathbb{N}$ be prime. Then p is relatively prime to every positive natural number except for p itself.

c. Let $a, b, c, n : \mathbb{N}$ and suppose that $3ab \equiv 3ac \mod n$. Then $b \equiv c \mod n$.

d. Let $a, b : \mathbb{N}$. Then $\gcd(a, b) = \gcd(a, \gcd(a, b))$.

e. Let $a, b : \mathbb{N}$. Then $\gcd(1 + a, 1 + b) = 1 + \gcd(a, b)$.

f. Let $n : \mathbb{N}$ and suppose n is not divisible by 3. Then $n^2 \equiv 1 \mod 3$.

Problem 2

Normally explorers are not allowed to wander around Jurassic Park unsupervised. But 10 brave CS22 TAs have left the normal tourist paths. Incredibly, they have stumbled onto a nest that is full of dinosaur eggs!

a. Using their deep knowledge of dinosaurs from TA camp, they determine that each egg is either a tyrannosaurus or brontosaurus egg, and there are twice as many brontosaurus eggs as there are tyrannosaurus eggs.

   The TAs decide to divide the tyrannosaurus eggs between themselves such that TA number n gets t_n tyrannosaurus eggs. For fairness and number-theoretic reasons, they require that for each pair of TAs m and n, 10 does not divide $t_m - t_n$.

   Is it possible for them to distribute the brontosaurus eggs with the same restriction? Why or why not?

b. The TAs eventually deliver the eggs to Rob, who decides to play a game. He arranges the eggs into three rows: the first row has 51 eggs, the second has 49, and the third has 5. In each move of this game, he can combine any two rows into one row, or he can split a row with $2n$ eggs into two rows each with n eggs. (This second move, of course, only works on rows with an even number of eggs.)

   Rob's goal is to create 105 rows, each with one single egg. Can he achieve this goal, or will he end the day disappointed? Justify your answer!

HINT:
Think about Rob's possible first moves; then remember that we've been talking a lot about greatest common divisors!

Problem 3

It's spring break! Time to introduce a fun new game to play with a friend, while you're bored and missing cs22.

On a piece of paper, write down two natural numbers $m, n > 0$.

The two of you will take turns following this rule: choose two (distinct) numbers that are written on the paper, and write down the difference of those two numbers. This should be positive (subtract the smaller one from the bigger one), and you can't repeat numbers that are already on the page.

You take the first turn, followed by your friend. Eventually someone will get stuck, unable to write down a new number. That person loses the game.

a. Prove that every number written down on the page is divisible by $\gcd(m, n)$.

b. Prove that all of the (positive) multiples of $\gcd(m, n)$ up to $\max(m, n)$ must be written down on the page by the time the game is over.

c. Your friend is very confident: they tell you that they can choose infinitely many pairs of starting numbers m and n that guarantee them a win. But you can do the same! Describe how you could choose pairs of numbers that guarantee you, the first player, will win the game.

Problem 4 (Mind Bender — *Extra Credit*)

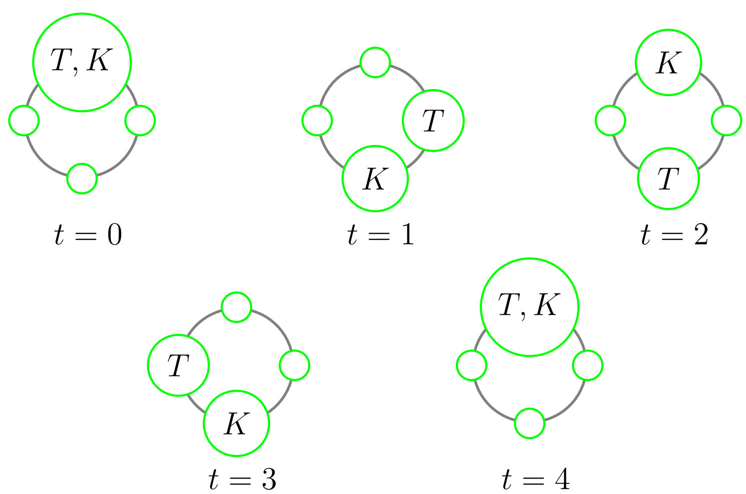
Fix a value $k \in \mathbb{N}^+$.

A t-rex and a kotasaurus are playing a game that involves hopping around a circular track of lily pads.[1] At the start, they stand on the same lily pad. Then, every second, the slow t-rex jumps one lily pad clockwise, while the swift kotasaurus jumps k lily pads clockwise. They keep hopping until they once again end up on the same lily pad as each other (regardless of whether it is the lily pad on which they started).

If there are n lily pads, where n is a positive natural number, determine, with proof, the number of seconds it will take for the t-rex and kotasaurus to finish their game.

You may cite without proof the *coprime divisibility lemma*: for any integers a, b, and c, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

As an example, here's how the game would go on a track of four lily pads if the kotasaurus can jump two lily pads per second (t is the number of seconds elapsed, and T represents the t-rex and K the kotasaurus):



[1] These are some *very* resilient lily pads.

HINT: One possible approach to this problem involves proving Lemma 1 below and using Lemma 1 to prove Lemma 2:

Lemma 1: For any integers n and k, we have $\gcd\left(\frac{n}{\gcd(n,k)}, \frac{k}{\gcd(n,k)}\right) = 1$.

Lemma 2: Let n be a positive natural number. Fix an integer k. If $s \in \mathbb{Z}$ is a solution to the congruence $kx \equiv 0 \pmod{n}$, then $\frac{n}{\gcd(n,k)} \mid s$.